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EXSAMEN 1. XII. 2012

(1) $$\begin{cases} y'' + y' = t\,u(t-2) \\ y(0) = 0, \quad y'(0) = 0 \end{cases}$$

$$t\,u(t-2) = (t-2)u(t-2) + 2u(t-2)$$

$$\mathcal{L}\{t\,u(t-2)\} = e^{-2s}\frac{1}{s^2} + e^{-2s}\frac{2}{s}$$

$$(s^2+s)Y(s) = e^{-2s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$$

$$Y(s) = \frac{2s+1}{s^3(1+s)}e^{-2s}$$

$$\frac{2s+1}{s^3(1+s)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{s+1}$$

$$= \frac{1}{s^3} + \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}$$

$$\frac{t^2}{2!} \qquad t \qquad -1 \qquad e^{-t}$$

ANSWER:

$$y(t) = u(t-2)\left\{\frac{(t-2)^2}{2} + t - 2 - 1 + e^{-(t-2)}\right\}.$$

(2) $u(x,y) = x^3 - 3xy^2$

In fact, $\nabla^2 u = 0$. Now

$$\begin{cases} \dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} = 3x^2 - 3y^2 \\[2ex] \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \end{cases}$$

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$$v(x,y) = 3x^2 y - y^3 + C(x)$$

$$\frac{\partial v}{\partial x} = \underline{6xy + C'(x)} = -\frac{\partial u}{\partial y} = \underline{6xy}$$

$$\Longleftrightarrow C'(x) = 0$$

<u>Answer:</u> $v(x,y) = 3x^2 y - y^3 + \text{Const.}$

$$\left(f(z) = z^3 = (x+iy)^3 = \cdots \right)$$

(3a) $f(x)$ is an odd function. Hence

$$a_n = 0, \quad n = 0, 1, 2, 3, \ldots$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx \overset{!}{=} \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} f(x) \sin(nx)\, dx$$

$$= \frac{2}{\pi} \cdot \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} x \sin(nx)\, dx$$

$$= \frac{4}{\pi^2} \left\{ \underbrace{\Big|_{-\pi/2}^{\pi/2} -x \frac{\cos(nx)}{n}}_{=0} + \int_{-\pi/2}^{\pi/2} \frac{\cos(nx)}{n}\, dx \right\}$$

$$= + \frac{4}{n^2\pi^2} \Big|_{-\pi/2}^{\pi/2} \sin(nx) = \frac{8}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$$b_{2n} = 0, \quad n = 2m-1 \quad \text{(odd index)}$$

$$b_{2m-1} = \frac{8}{(2m-1)^2\pi^2} (-1)^{m+1}$$

$$f(x) = \frac{8}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)^2} \sin[(2m-1)x]$$

(3b) Notice that $f(x)$ is the same function as in example 3a above.
Separation of variables:

$$\frac{\partial u}{\partial t} = 9 \frac{\partial^2 u}{\partial t^2}, \quad u(x,t) = X(x)\,T(t)$$

$$X\dot{T} = 9 X'' T$$

$$\frac{X''(x)}{X(x)} = \frac{\dot{T}(t)}{gT(t)} = -\lambda \quad (\text{Constant of separation})$$

$$\begin{cases} X'' + \lambda X = 0 & X(0) = 0 = X(\pi). \\ \dot{T} + g\lambda T = 0 & \end{cases}$$

Only the functions

$$X_n(x) = B_n \sin(nx), \quad \lambda = n^2$$

will do. We obtain the solutions

$$B_n e^{-gn^2 t} \sin(nx), \quad n = \pm 1, \pm 2, \ldots$$

($n<0$ are absorbed)

By superposition

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-gn^2 t} \sin(nx)$$

$$f(x) \stackrel{?}{=} u(x,0) = \sum_{n=1}^{\infty} B_n \sin(nx)$$

This is a Fourier sine series and the coefficients are as in 3a.

SOLUTION: $$u(x,t) = \frac{8}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} e^{-g(2m-1)^2 t}}{(2m-1)^2} \sin[(2m-1)x]$$

④ $$y(t) + 4\int_0^t (t-\tau)\,y(\tau)\,d\tau = 2$$

The integral is the convolution of $f(t) = t$ and $y(t)$. Taking the Laplace transform we get

$$Y(s) + 4\underbrace{\mathcal{L}\{t\}}_{1/s^2}\,Y(s) = \frac{2}{s}$$

$$Y(s)\left[1 + 4/s^2\right] = \frac{2}{s}$$

$$Y(s) = \frac{2s}{s^2 + 2^2}\ , \qquad y(t) = 2\cos(2t)$$

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⑤ $$\hat{g}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)\,e^{-i\omega x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^1 (1-x)\,e^{-i\omega x}\,dx = \frac{1}{\sqrt{2\pi}}\left.\frac{e^{-i\omega x}}{-i\omega}\right|_0^1$$

$$+ \frac{1}{\sqrt{2\pi}}\left.\frac{xe^{-i\omega x}}{i\omega}\right|_0^1 - \frac{1}{\sqrt{2\pi}}\int_0^1 \frac{e^{-i\omega x}}{i\omega} = \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{i\omega}$$

$$+ \frac{1}{\sqrt{2\pi}}\left.\frac{e^{-i\omega x}}{-\omega^2}\right|_0^1 = \frac{1}{\sqrt{2\pi}}\left\{\frac{1}{i\omega} + \frac{1 - e^{-i\omega}}{\omega^2}\right\}$$

$$⑥ \quad I = \int_0^{2\pi} \frac{\cos^2(\theta)}{2+\sin(\theta)}\, d\theta = \oint_{|z|=1} \frac{\left(\frac{z+\frac{1}{z}}{2}\right)^2}{2+\frac{z-\frac{1}{z}}{2i}} \frac{dz}{iz}$$

The integrand can be written as

$$F(z) = \frac{z^4+2z^2+1}{2z^2(z^2+4iz-1)} = \frac{(z^2+1)^2}{2z^2(z-(\sqrt{3}-2)i)(z+(\sqrt{3}+2)i)}$$

$$z^2+4iz-1 = 0 \iff \begin{cases} z = i(-2+\sqrt{3}) & \text{(included)} \\ z = -(2+\sqrt{3})i & \text{(outside the contour)} \end{cases}$$

Inside the unit circle we have the simple pole $(\sqrt{3}-2)i$ and the double pole $z=0$. Thus

$$I = 2\pi i \left\{ \operatorname*{Res}_{z=0} F(z) + \operatorname*{Res}_{z=(\sqrt{3}-2)i} F(z) \right\}$$

i) <u>The Residue at $z = (\sqrt{3}-2)i$</u>.

1°) Multiply by $z-(\sqrt{3}-2)i$,
2°) Evaluate at $z=(\sqrt{3}-2)i$.

$$z = i(-2+\sqrt{3})$$

$$z^2 = -7+4\sqrt{3}, \quad z^2+1 = -6+4\sqrt{3}$$

$$(z^2+1)^2 = 12\{7-4\sqrt{3}\}$$

$$\operatorname*{Res}_{(-2+\sqrt{3})i} F(z) = \left.\frac{(z^2+1)^2}{2z^2(z+(\sqrt{3}+2)i)}\right|_{z=(-2+\sqrt{3})i} = \sqrt{3}\, i$$

ii) The Residue at $z=0$ (double pole):

$$\frac{d}{dz}\left[\frac{z^4+2z^2+1}{2(z^2+4iz-1)}\right]$$

1°) Multiply by $(z-0)^2$,
2°) Differentiate once,
3°) Evaluate at $z=0$.

$$= \frac{(4z^3+4z)(z^2+4iz-1)-(2z+4i)(z^4+2z^2+1)}{2(z^2+4iz-1)^2}$$

$$= \frac{-4i}{2(-1)^2} = -2i \quad \text{at } z=0$$

This is the residue. By the residue theorem

iii)

ANSWER:

$$\int_0^{2\pi} \frac{\cos^2(\theta)}{2+\sin(\theta)}\,d\theta = 2\pi i\left\{\sqrt{3}\,i-2i\right\}$$
$$= (2-\sqrt{3})\,2\pi.$$

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(7) $$f(z) = \frac{e^z-1-z}{z^2(z-1)(z+1)} = \frac{z^2/2! + \frac{z^3}{3!} + \cdots}{z^2(z-1)(z+1)}$$

Hence $\lim_{z\to 0} f(z) = \frac{1/2!}{(-1)(1)} = -\frac{1}{2}$ $(\neq \infty)$ and so $z=0$ is not a pole. (Removable singularity).

$z=1$ is a simple pole $(e^1-1-1\neq 0)$ with residue
$$\operatorname*{Res}_{z=1}\{f(z)\} = \frac{e-2}{2}$$

$z=-1$ is a simple pole $(e^{-1}-1+1\neq 0)$ with residue
$$\operatorname*{Res}_{z=-1}\{f(z)\} = \frac{e^{-1}}{-2} = -\frac{1}{2e}.$$