Problem 1 Let $T > 0$ and consider the transport equation

$$(1) \qquad Lu(x,t) := \Big(\frac{\partial}{\partial t} + (\tfrac{2}{3} - x)\frac{\partial}{\partial x} + 5\Big)u(x,t) = 0 \qquad \text{in} \qquad (0,1)\times(0,T).$$

We solve this problem with the explicit upwind finite difference scheme:

$$(2) \qquad L_h U_m^n := \Big(\frac{\Delta_k}{k} + (\tfrac{2}{3} - x_m)^+ \frac{\nabla_h}{h} - (\tfrac{2}{3} - x_m)^- \frac{\Delta_h}{h} + 5\Big)U_m^{n-1} = 0 \quad \text{in} \quad \mathbb{G},$$

where $a^\pm = \max(\pm a, 0)$, $a = a^+ - a^-$, $|a| = a^+ + a^-$, and the grid is defined by $M, N \in \mathbb{N}$, $h = \frac{1}{M}$, $k = \frac{T}{N}$, and

$$\overline{\mathbb{G}} = \{P = (x_m, t_n) = (mh, nk) : m = 0, \dots, M;\ n = 0, \dots, N\}.$$

a) Show that the scheme (2) is a positive coefficient scheme under a CFL type of condition. Compute the constant in the CFL condition.

b) Find the local truncation error τ of (2), where

$$\tau_m^n = Lu_m^n - L_h u_m^n \qquad \text{for} \qquad m = 1, \dots, M-1,\ n = 1, \dots, N,$$

for any smooth function u and where $u_m^n = u(x_m, t_n)$.

Hint: In (2) most terms are taken at $t = t_{n-1}$! Taylor expand u *and* u_x.

Let $\overline{Q}_T = [0,1]\times[0,T]$ and $\partial^*\mathbb{G}$ be the part of the space-time boundary of $\mathbb{G}$ that can be reached by the scheme (2).

c) Determine whether $x = 0$ and/or $x = 1$ are inflow boundaries. Explain on what part $\partial^* Q_T$ of the space-time boundary of Q_T we must impose initial and boundary conditions.

Give the two stencils of the scheme (2) and explain why

$$\partial^*\mathbb{G} = \{x = 0\}\times(0,T) \bigcup \{x = 1\}\times(0,T) \bigcup [0,1]\times\{t = 0\}.$$

We assume that a CFL condition holds so the scheme (2) is monotone. You may therefore use without proof that it is stable with respect to the right hand side:

$$L_h V_P = F_P \text{ for } P \in \mathbb{G}, \quad V_P = 0 \text{ for } P \in \partial^*\mathbb{G} \qquad \Longrightarrow \qquad \max_{P\in\overline{\mathbb{G}}} |V_P| \le \frac{1}{5} \max_{P\in\overline{\mathbb{G}}} |F_P|.$$

d) Let u and U solve (1) and (2) with the same initial and Dirichlet boundary conditions. Show an error bound of the form

$$\max_{(x_m,t_n)\in\mathbb{G}} |u_m^n - U_m^n| \le C_1 h + C_2 k.$$

Then show that $C_1 \le \frac{2}{15}\|u_{xx}\|_{L^\infty(0,1)}$ and find an expression of C_2 only in terms numerical constants and norms of derivatives of u.

Problem 2 Consider the boundary value problem

(3) $$-Lu := -u_{xx} - (1+x)u_{yy} = 0 \qquad \text{in} \qquad \Omega,$$

(4) $$\begin{cases} u(0,y) = 0 \\ u(x,0) = 0 \end{cases} \qquad \text{for} \qquad (0,y), (x,0) \in \partial\Omega,$$
$$u(x,y) = 1 \qquad \text{for} \qquad (x,y) \in \partial\Omega,\ x,y > 0.$$

where the domain is the part of the unit square laying inside the circle $x^2+y^2 = \frac{10}{9}$:

$$Q_1 := [0,1] \times [0,1], \qquad \Omega := \Big\{(x,y) \in Q_1 \ : \ x^2+y^2 \le \tfrac{10}{9}\Big\}.$$

Let $\overline{\mathbb{G}}$ be an equidistant grid on Q_1 with step size $h = \frac{1}{3}$:

$$\overline{\mathbb{G}} := \Big\{(x_i, y_j) = h(i,j) \ : \ i,j = 0,1,2,3\Big\}.$$

Note that $x_1^2 + y_3^2 = \frac{10}{9} = x_3^2 + y_1^2$. Using the method of *fattening the boundary* combined with central finite differences, we approximate this problem on a subgrid $\overline{\mathbb{G}}_1 \subset \overline{\mathbb{G}}$. Let $U(P)$ for $P \in \overline{\mathbb{G}}_1$ be solution of the resulting scheme.

a) Write down the stencil for the scheme and make a sketch of the whole grid $\overline{\mathbb{G}}_1$ where you indicate the fattened boundary.

Give a row-wise enumeration of $\overline{\mathbb{G}}_1$ starting from the origin, where you list the node numbers and corresponding coordinates. Which nodes P are boundary nodes, $P \in \partial\mathbb{G}_1$?

What boundary condition should be imposed on $U(P)$ for every $P \in \partial\mathbb{G}_1$?

b) Let $\vec{U} = (U(P))_{P\in\mathbb{G}_1}$ be the vector of U-values at interior nodes in increasing order in the enumeration from a). Find a matrix A and a vector $\vec{b}$ such that

$$A\vec{U} = \vec{b} \qquad \text{where} \qquad A = \begin{pmatrix} \frac{14}{3} & -1 & -\frac{4}{3} & 0 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}.$$

Problem 3 Consider the following variational problem/weak formulation:

(5) Find $u \in V$ such that $a(u,v) = F(v) \quad \forall v \in V$,

where $V := \{v \in H^1(0,1) : v(1) = 0\}$ is a subspace of $H^1(0,1)$,

$$a(u,v) = \int_0^1 \Big[7u_x(x)v_x(x) - 5u(x)v_x(x)\Big]\,dx, \qquad \text{and} \qquad F(v) = 3v(0).$$

a) Find what PDE and boundary value problem (5) is the weak formulation of.

Show that a is continuous and coersive on $V \times V$.

Hint: You may use that $\|v\|_{L^2} \leq \|v_x\|_{L^2}$ for $v \in V$.

b) Approximate problem (5) using the $\mathbb{P}_1$ finite element method on a uniform grid $x_i = ih$, $i = 0, ..., M$ with $h = 1/M$. Show that this method can be expressed as a linear system

$$A\vec{U} = \vec{F}.$$

Compute the stiffness matrix A and the load vector $\vec{F}$ for arbitrary M.