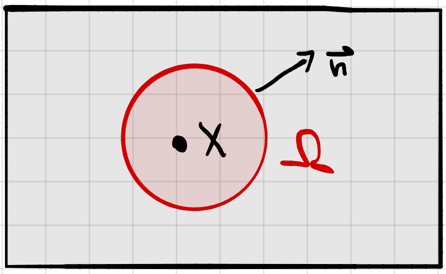
1. Derivation of the heat equation

3 Steps involved:

a) Conservation of energy:



- $e(\vec{x},t)$ (with units $[J/m^3]$) density of internal energy at Ω, point x $[m]$ and time t $[s]$
- $\vec{F} = \vec{F}(\vec{x},t)$ $[J/(m^2 s)]$ a vector field describing the heat flux through some surface $\partial\Omega$
- $p = p(\vec{x},t)$ $[J/(m^3 s)]$ a scalar function describing the power density of a heat source (think of candle in the room)

Then the principle of conservation of energy dictates that

$$\underbrace{\frac{d}{dt}\int_{\Omega} e(\vec{x},t)\,dV}_{\text{Total change of energy}} = \underbrace{-\int_{\partial\Omega} \vec{F}\cdot\vec{n}\,dS}_{\text{total energy flowing through surface per time unit}} + \underbrace{\int_{\Omega} p\,dV}_{\text{total energy generated inside } \Omega \text{ per time unit}}$$

Now recall the Gauß / divergence theorem

$$\int_{\partial\Omega} \vec{F}\cdot\vec{n}\, dS = \int_{\Omega} \operatorname{div} \vec{F}\, dV$$, plugging this in yields

$$\frac{d}{dt}\int_{\Omega} e(\vec{x},t)\,dV = \int_{\Omega} \partial_t e(\vec{x},t)\,dV = \int_{\Omega}(-\operatorname{div}\vec{F} + p)\,dV$$

Since this apply to any arbitrary domain Ω, it must hold that

$$\partial_t e(\vec{x},t) + \operatorname{div}\vec{F}(\vec{x},t) = p(\vec{x},t) \quad \text{for all } x\in\Omega_0,\ \text{and } t>0.$$

b) Constitutive laws:

Absolute temperature $T(\vec{x},t)$ [K] is a measure for the storage of energy at point $\vec{x}$ and time t.

- First constitutive law relates internal energy density to temperature:

$$e = e_0 + \sigma(T - T_0)$$

chosen reference temperature

$$= e_0 + \sigma\vartheta \qquad (\vartheta = T - T_0)$$

$\sigma = \sigma(\vec{x})$ $[J/(m^3 K)]$ specific heat capacity

- Second constitutive law (Fourier's law) relates temperature to the heat flux:

$$\vec{F} = -\lambda\nabla\vartheta$$

- $\lambda = \lambda(x)$ $[J/(m\,K\,s)]$ is the heat conductivity
- If additional energy transport through convection (e.g. in a fluid) occurs: $\vec{F} = -\lambda\nabla\vartheta + \vec{v}e$.

Plugging in constitutive laws gives the heat equation:

$$\sigma \partial_t \vartheta - \nabla \cdot (\lambda \nabla \vartheta) = p \quad \forall x \in \Omega_0, t > 0.$$

If we assume that λ does not depend on $\vec{x}$ (e.g. for homogeneous material), then this reduces to

$$\sigma \partial_t \vartheta - \lambda \Delta \vartheta = p \quad \forall x \in \Omega_0, t > 0.$$

c) Boundary and initial conditions

To finally determine the temperature ϑ, we must also know:

- the initial temperature distribution $\vartheta_0(x) = \vartheta(x,0)$
- whether and what kind of energy exchange occurs with the boundary $\partial\Omega_0$ $\Rightarrow$ boundary conditions.

Most common boundary conditions (for the heat equation)

- $\vec{F} \cdot \vec{n} = \kappa(\vec{x}, t)(\vartheta - \vartheta_a)$

  ϑ_a is some ambient temperature, κ $[J/(m^2 s K)]$ is the heat transfer coefficient.

  With Fourier's law this gives

  $$-\lambda \nabla \vartheta \cdot n = -\lambda \partial_n \vartheta = \kappa(\vartheta - \vartheta_a)$$

  Robin boundary condition:

  $$\lambda \partial_n \vartheta + \kappa(\vartheta - \vartheta_a) = 0 \text{ on } \partial\Omega_0.$$

- Limit case $\mathcal{K} = 0$ (perfectly isolated):

(Homogeneous) Neumann condition: $\lambda \partial_n \vartheta = 0$ on $\partial \Omega_0$

- Limit case $\mathcal{K} \to \infty$ (infinitely fast heat exchange):

(Inhomogeneous) Dirichlet condition: $\vartheta = \vartheta_a$ on $\partial \Omega_0$.

- From now on we will consider the heat equation in the form

$$\partial_t u = u_t - \overset{:= \lambda}{c^2} \Delta u = \rho.$$

2. Solution on bounded intervals; Separation of variables

Want to compute temperature of a rod.

$$\begin{cases} \partial_t u = c^2 \partial_{xx} u \\ u(0,t) = u(L,t) = 0 & \text{Dirichlet boundary condition} \\ u(x,0) = f(x) & \text{initial condition} \end{cases}$$

- Separation of variables: Maybe our solution $u(x,t)$ can be written as a product of a function $F(x)$ and $G(t)$?

Ansatz: $u(x,t) = F(x)G(t)$, then

$$\partial_t u = G'(t)F(x) = c^2 G(t) F''(x)$$

Sorting terms:
$$\underbrace{\frac{G'(t)}{c^2 G(t)}}_{\text{function in } t} = \underbrace{\frac{F''(x)}{F(x)}}_{\text{function in } x}$$

Since x and t can vary independent of each other, we must have that

$$\frac{G'(t)}{c^2 G(t)} = k = \frac{F''(x)}{F(x)}$$

for some constant k which we don't know yet.

Thus we get

$$F''(x) + k F(x) = 0$$
$$g'(t) + c^2 k g(t) = 0$$

Let's have a look at k and F first.

- **Case $k = 0$:**

Then $F''(x) = 0 \Rightarrow F(x) = A + Bx$ for some constants A, B.

B.c. gives $F(0) g(t) = 0$ so either $g(t) = 0 \ \forall t$ (which is uninteresting since $u(x,t) = 0 \ \forall x, t$) or $F(0) = 0$.

If $F(0) = 0 \Rightarrow A = 0$

Second b.c gives $F(L) g(t) = 0$, same reasoning as before gives $F(L) = BL = 0 \Rightarrow 0$, so $F(x) = 0$ if $k = 0$.

That is dull :(.

- **Case $k < 0$:** Then $F'' + kF$ has the general solution

$$F(x) = A e^{\sqrt{-k}\,x} + B e^{-\sqrt{-k}\,x}.$$

Using our boundary conditions again and excluding $g(t) \equiv 0$, we get that

$$\begin{aligned} F(0) &= A + B = 0 \\ F(L) &= A e^{\sqrt{-k}\,L} + B e^{-\sqrt{-k}\,L} = 0 \end{aligned} \iff \underbrace{\begin{pmatrix} 1 & 1 \\ e^{\sqrt{-k}\,L} & e^{-\sqrt{-k}\,L} \end{pmatrix}}_{=:\, C} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\det C = e^{-\sqrt{-k}\,L} - e^{\sqrt{-k}\,L} \begin{cases} = 0 & \text{if } k = 0 \\ < 0 & \text{if } k < 0. \end{cases}$$

So for $k < 0$, matrix C is invertible $\Rightarrow A = B = 0$.

This is a very unexciting solution.

• Case $\lambda > 0$. Still our general solution is

$$F(x) = \tilde{A} e^{\sqrt{-\lambda}x} + \tilde{B} e^{-\sqrt{-\lambda}x} \qquad \text{but now}$$

$$= \tilde{A} e^{i\sqrt{\lambda}x} + \tilde{B} e^{-i\sqrt{\lambda}x}$$

so in general our solution F can now written as

$$F(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x.$$

• Using b.c. $u(0,t) = 0$ we get
$F(0) = A = 0.$

• Using b.c. $u(L,t) = 0$ we get

$$F(L) = B\sin\sqrt{\lambda}L = 0$$

This we can satisfy for some non-trivial B if $\sqrt{\lambda}L$ is some integer multiple of $n\pi$:

$$\sqrt{\lambda}L = n\pi \iff \lambda = \left(\frac{n\pi}{L}\right)^2.$$

• Enough to consider $n > 0$, otherwise sin just changes sign and can be incorporated into B.

• Now we can turn to $G(t)$:

$$G'(t) + c^2\lambda G(t) = 0$$

For each n will we get an equation

$$G_n'(t) + \left(\frac{cn\pi}{L}\right)^2 G(t) = 0$$

$$\Rightarrow \quad G_n(t) = A_n e^{-\left(\frac{cn\pi}{L}\right)^2 t}$$

So starting from the separation of variables ansatz and incorporating b.c. and ignoring trivial / uninteresting solutions we get possible solution functions of the form

$$u_n(x,t) = F(x) G_n(t) = A_n e^{-\left(\frac{cn\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right).$$

Of course any superimposition / linear combination will be a solution to the heat equation as well, and will satisfy our boundary condition:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{cn\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

- How can we determine A_n? Use initial condition!

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)$$

That smells suspiciously like a Fourier series (More precisely a Sin series).

- So A_n is the Fourier coefficient to the odd extension f_o of f!

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx.$$

We summarize our derivation in the following theorem.

Theorem 1

The heat equation problem

$$\begin{cases} \partial_t u - c^2 \partial_{xx} u = 0 \\ u(0,t) = u(L,t) = 0 & \text{Homogeneous Dirichlet conditio} \\ u(x,0) = f(x) & \text{Inhomogeneous initial conditions} \end{cases}$$

is solved by

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{cn\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L} x\right)}$$

with

$$\boxed{A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) .}$$

3. Solution on an infinite rod

- $\partial_t u = c^2 \partial_x^2 u$ (1)
- Boundary conditions: $\lim_{x \to \pm\infty} u(x,t) = 0.$
- Initial conditions $u(x,0) = f(x).$

Idea: Fourier series useful for bounded interval, maybe the Fourier transform might be helpful now?

- Start with Fourier transform (1) in x:

$$\mathcal{F}(\partial_t u) = c^2 \mathcal{F}(\partial_x^2 u)$$

- $$\mathcal{F}(\partial_t u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \partial_t u(x,t) e^{-iwx} dx$$
$$= \frac{\partial}{\partial t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-iwx} dx$$
$$= \frac{\partial}{\partial t} \hat{u}(w,t).$$

- $$\mathcal{F}(\partial_{xx} u) = iw\, \mathcal{F}(\partial_x u) = -w^2 \mathcal{F}(u).$$

So we obtain $$\partial_t \hat{u}(w,t) = -c^2 w^2 \hat{u}(w,t).$$

Freezing w, we can treat this as an ODE.

$$\hat{u}(w,t) = A(w) e^{-c^2 w^2 t}.$$

Fourier transform the initial condition gives
$$\hat{f}(w) = \hat{u}(w,0) = A(w).$$

Thus

$$u(x,t) = \mathcal{F}_w^{-1}(\hat{u}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{-c^2w^2t} e^{iwx} \, dw$$

- Now recall the convolution theorem for $\mathcal{F}$, which we can reformulate as

$$\frac{1}{\sqrt{2\pi}} f * g = \mathcal{F}^{-1}(\hat{f}\hat{g}).$$

so

$$u(x,t) = \frac{1}{\sqrt{2\pi}} f * \boxed{\tilde{\mathcal{F}}_w^{-1}\left(e^{-c^2w^2t}\right).}$$

- Can we compute the last term? Now remember from lecture 10 that

$$\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}}.$$

so to compute $\square$ we set $\frac{1}{4a} = c^2t \iff a = \frac{1}{4c^2t}$ and then we obtain that

$$\square = \frac{\sqrt{2}}{c\sqrt{4t}} \cdot e^{-\frac{x^2}{4c^2t}}$$ and thus

$$u(x,t) = f * \left(\frac{1}{c\sqrt{4\pi t}} e^{-\frac{x^2}{4c^2t}} \right)$$

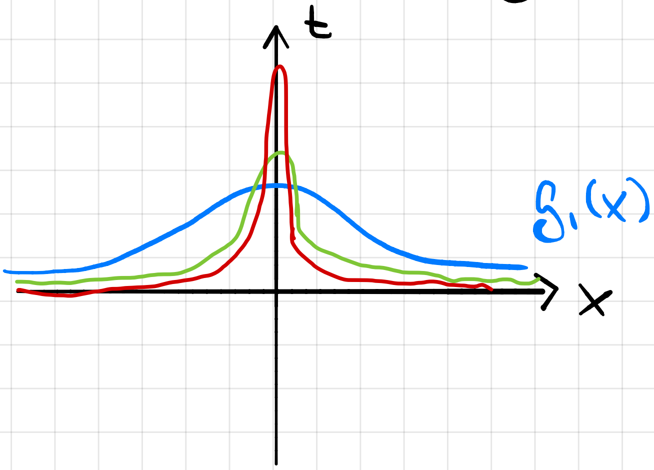
$$= \int_{-\infty}^{\infty} f(\sigma) \frac{1}{c\sqrt{4\pi t}} e^{-\frac{(x-\sigma)^2}{4c^2t}} \, d\sigma.$$

Define Heat kernel by

$$G_t(x) = G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

then $u(x,t) = (f * G_{c^2t})(x)$.

- Observations:
  - $\int_{-\infty}^{\infty} S(x,t)\, dx = 1 \quad \forall t > 0.$
  - For $t \to 0$ $\quad S(x,t) \to \delta$ (Dirac function)



  - $S(x,t)$ solves the heat equation

- Same applies to $S(x, c^2 t)$. Thus we expect that

$$u(x,t) = (f * S_{c^2t})(x) \longrightarrow (f * \delta)(x) = f(x).$$

- That is indeed the case but we won't prove it rigorously.

Theorem 2

The heat equation on x-axis with initial conditions

$$u(x,0) = f(x)$$

can be solved by

$$\boxed{u(x,t) = (f * S_{c^2t})(x) = \int_{-\infty}^{\infty} f(v) \frac{1}{c\sqrt{4\pi t}} e^{-\frac{(x-v)^2}{4c^2t}}},$$

and we have that

$$\lim_{t \to 0} u(x,t) = f(x).$$

4. Laplace equation

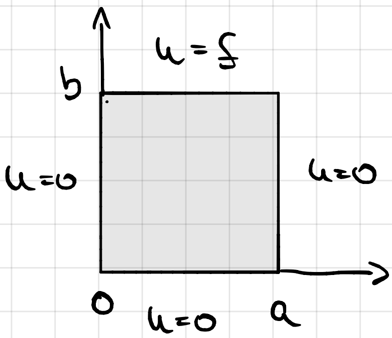
- For an equilibrium state, temperature will remain steady / not change over time any more $\Rightarrow \partial_t u(x,t) = 0$ and thus the temperature field u at equilibrium satisfies the Laplace problem

$$c^2 \Delta u = 0.$$

- If the right-hand side is not 0, we talk about the Poisson problem.
- Consider now the Laplace problem in 2D:

$$\partial_x^2 u + \partial_y^2 u = 0 \qquad + \text{ boundary condition}$$

4.1 Solution on a bounded rectangular domain



- $\Omega = [0,a] \times [0,b]$
- $u(x,0) = u(0,y) = u(a,y) = 0$
  $u(x,b) = f(x)$

- Idea: Use separation of variables again

Ansatz: $u(x,y) = F(x)G(y)$

$0 = \Delta u = F''(x)G(y) + F(x)G''(y)$ and as in yesterday's lecture

$$\Rightarrow \quad -\frac{F''(x)}{F(x)} = \frac{G''(y)}{G(y)} = k \text{ for some constant } k.$$

So we need to solve

$$F''(x) + k\,F(x) = 0 \qquad 1)$$
$$G''(y) - k\,G(y) = 0 \qquad 2)$$

- To find possible solutions to 1) we proceed as in yesterday's lecture:

$$k = \left(\frac{n\pi}{a}\right)^2 \quad n \in \mathbb{N}, \qquad F(x) = \sin \frac{n\pi}{a} x \qquad n \in \mathbb{N}.$$

- The general solution for 2) is

$$G_n(y) = \tilde{A}_n e^{-\frac{n\pi}{a} y} + \tilde{B}_n e^{\frac{n\pi}{a} y}$$
$$= A_n \sinh\left(\frac{n\pi}{a} y\right) + B_n \cosh\left(\frac{n\pi}{a} y\right).$$

- General solution which satisfies $u(0,y) = u(a,y) = 0$:

$$u(x,y) = F(x)\,G_n(y) = \left(A_n \sinh\left(\frac{n\pi}{a} y\right) + B_n \cosh\left(\frac{n\pi}{a} y\right)\right) \sin \frac{n\pi}{a} x.$$

- Requiring that $u(x,0) = 0$ leads to

$$A_n \sinh(0) + B_n \cosh(0) = 0 \Rightarrow B_n = 0.$$

- To satisfy $u(x,b) = f(x)$, we superimpose the solutions u_n

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{a} y\right) \sin\left(\frac{n\pi}{a} x\right)}$$

and then

$$f(x) = u(x,b) = \sum_{n=1}^{\infty} \boxed{A_n \sinh\left(\frac{n\pi}{a} b\right)} \sin\left(\frac{n\pi}{a} x\right) \quad (*)$$

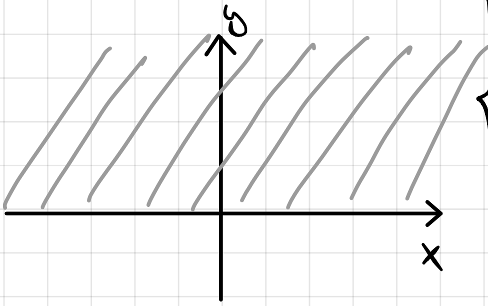
so similar to yesterday's derivation, $(*)$ must correspond to the Fourier coefficient of the odd extension of f and thus

$$\boxed{A_n = \left(\sinh \frac{n\pi b}{a}\right)^{-1} \cdot \frac{2}{a} \cdot \int_0^a f(x) \sin\left(\frac{n\pi}{a} x\right) dx}.$$

4.2 Solution in the half-plane

- $\Omega = \{ (x,y) \in \mathbb{R}^2 : y \geq 0 \}$

- Want to solve



$$\begin{cases} \partial_x^2 u + \partial_y^2 u = 0 & 1) \\ \lim\limits_{x \to \pm\infty} u(x,y) = \lim\limits_{y \to \infty} u(x,y) = 0 & 2) \\ u(x,0) = f(x) \text{ with } \lim\limits_{x \to \pm\infty} f(x) = 0. & 3) \end{cases}$$

- Use Fourier transform 1) w.r.t. x (we write $\mathcal{F}_x$):

$$\mathcal{F}_x(\partial_x^2 u) + \mathcal{F}_x(\partial_y^2 u) = \mathcal{F}(0) = 0$$

$$\Rightarrow \quad -\omega^2 \hat{u}(\omega, y) + \partial_y^2 \hat{u}(\omega, y) = 0$$

As before we treat this as an ODE in y and obtain

$$\hat{u}(\omega, y) = A(\omega) e^{-|\omega| y} + B(\omega) e^{|\omega| y}.$$

- Fourier transform of b.c. $\lim\limits_{\to \infty} u(x,y) = 0$ leads us to

$$\lim_{y \to \infty} \hat{u}(\omega, y) = 0 \Rightarrow B(\omega) = 0 \Rightarrow \hat{u}(\omega, y) = A(\omega) e^{-|\omega| y}$$

- Now transform $u(x,0) = f(x)$, thus

$$\hat{u}(\omega, 0) = \hat{f}(\omega) \Rightarrow A(\omega) = \hat{f}(\omega).$$

- Consequently we obtain that

$$\hat{u}(\omega, y) = \hat{f}(\omega) e^{-|\omega| y}$$

• Now take the inverse Fourier transform to obtain $u(x,y)$

$$u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w)\, e^{-|w|y}\, e^{iwx}\, dw$$

$$= \mathcal{F}^{-1}\left(\hat{f}(w)\, e^{-|w|y} \right)$$

By the convolution theorem stating that $(f*g)^{\wedge} = \sqrt{2\pi}\, \hat{f} \cdot \hat{g}$ we obtain

$$u(x,y) = \frac{1}{\sqrt{2\pi}} \cdot f * \mathcal{F}_x^{-1}\left(e^{-|w|y}\right)$$ (convolution w.r.t x variable)

• Now recall from lecture 10 we know that

$$\mathcal{F}\left(e^{-a|x|}\right) = \sqrt{\frac{2}{\pi}}\, \frac{a}{a^2+w^2}$$ and therefore

$$\mathcal{F}_x\left(\sqrt{\frac{2}{\pi}}\, \frac{y}{y^2+x^2} \right) = e^{-|w|y}$$ (Note that $\mathcal{F} = \mathcal{F}^{-1}$ for these two functions).

which shows that

$$\boxed{u(x,y) = \frac{1}{\sqrt{2\pi}}\, f * \left(\sqrt{\frac{2}{\pi}}\, \frac{y}{y^2+x^2} \right) = \int_{-\infty}^{\infty} f(t) \cdot \frac{1}{\pi}\, \frac{y}{y^2+(x-t)^2}\, dt .}$$

$$= \left(f * P_y\right)(x)$$

$$P_y(x) = \frac{1}{\pi}\, \frac{y}{y^2+x^2}$$ is the Poisson kernel for the upper half space.

Observations :

- $$\int_{-\infty}^{\infty} P_y(x)\,dx = 1 \quad \forall y > 0$$

- $$(\partial_x^2 + \partial_y^2)\, P_y(x) = 0$$

- $P_y(x) \longrightarrow \delta(x)$ for $y \longrightarrow 0$.

Thus we expect that

$$\lim_{y\to 0} u(x,y) = \lim_{y \to 0} (f * P_y)(x) = (f * \delta)(x) = f(x).$$

So u does indeed satisfy our boundary condition on the bottom part of Ω.