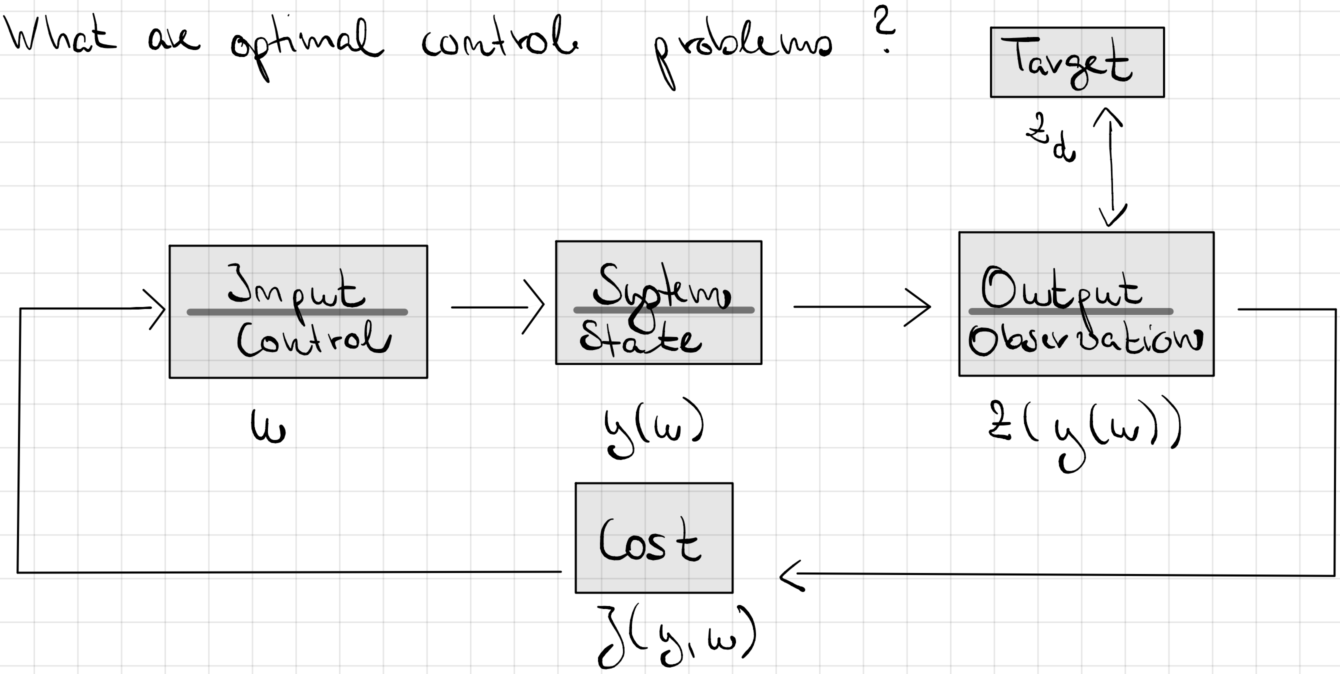
Lecture 01: Introduction to Optimization II

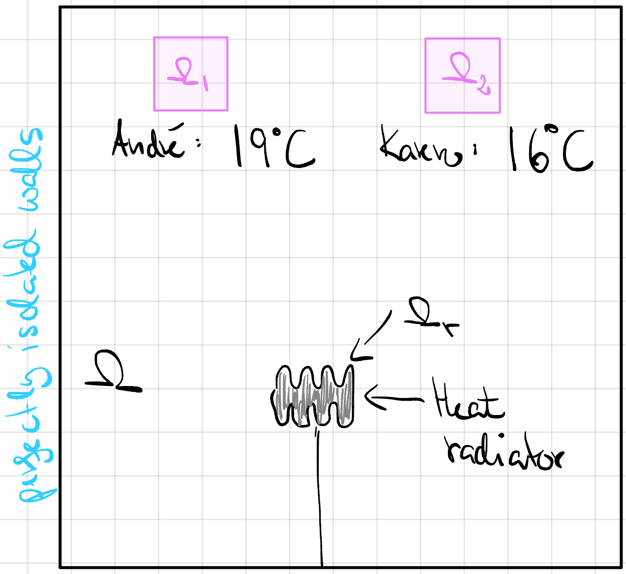
- Optimization II also known as:

"Optimal control of Partial Differential Equations (PDE)" or
"Optimal control problems (OCP) governed by PDEs" or
"Optimization with PDE constraints" or ...

- What are optimal control problems?



Example 1: Optimal heat source



- Input: radiator as heat source u
- State: Resulting temperature y in Ω
- Observation: temperature $y|_{\Omega_1 \cup \Omega_2} := z$
- Target / desired value:

$$z_d \quad \text{s.t.} \quad z_d(x) = \begin{cases} 19 & x \in \Omega_1 \\ 16 & x \in \Omega_2. \end{cases}$$

- Cost? E.g. the 'distance'

between output and target. But also heating costs?

$$J(y,u) = \text{'Distance' to target} + \text{heating costs}$$

Mathematically we want them to solve the following problem:

Minimize the cost functional

$$J(y,w) = \underbrace{\frac{1}{2} \int_{\Omega_1 \cup \Omega_2} |y(x) - z_d(x)|^2 \, dx}_{\text{measures distance to target}} + \underbrace{\frac{\alpha}{2} \int_{\Omega_r} w^2(x) \, dx}_{\text{measures heating costs}}$$

subject to the state problem

$$\begin{cases} -\Delta y = w & \text{in } \Omega \quad \text{stationary heat equation} \\ \dfrac{\partial y}{\partial n} = 0 & \text{on } \partial\Omega \quad \text{homogeneous Neumann condition} \end{cases}$$

state problem

This is an example of a linear-quadratic elliptic control problem

with distributed control

(functional)

w is a "volume"-based control.

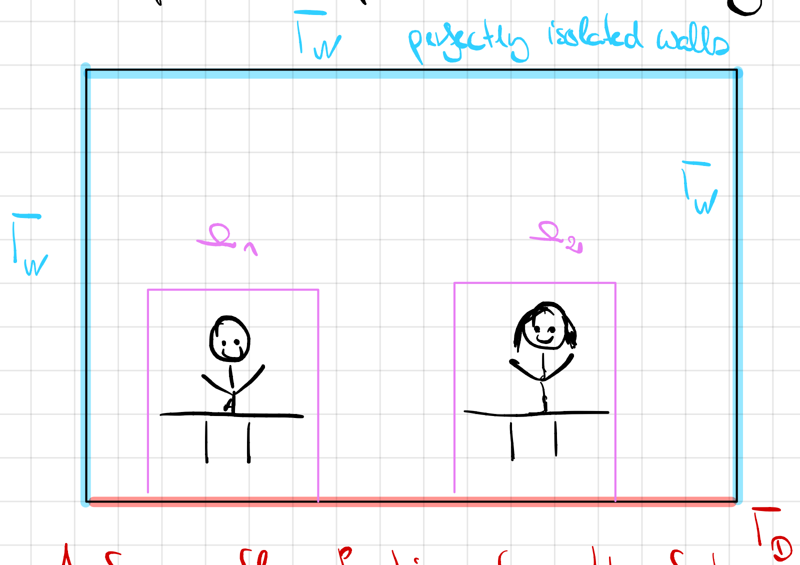
- Sometimes there are also constraints on the control or state.

E.g. the heat radiator has a minimal and maximal heat output:

$$0 \leq w(x) \leq w_{max} \qquad x \in \Omega_r .$$

This is an example of pointwise control constraints, more specifically box constraints for the control.

Example 2: Optimal floor heating in 3D



$$z_d(x) = \begin{cases} 19 & x \in \Omega_1 \\ 16 & x \in \Omega_2. \end{cases}$$

A fancy floor heating from the future which offers pointwise controllable heating!

Now minimize

$$J(y,w) = \frac{1}{2} \int_{\Omega_1 \cup \Omega_2} |y(x) - z_d(x)|^2 + \frac{\alpha}{2} \int_{\Gamma_D} w^2(x)\, d\Gamma$$

where y is given by the state problem

$$\begin{cases} -\Delta y = f & \text{in } \Omega \\ \frac{\partial y}{\partial n} = 0 & \text{on } \Gamma_W \\ y = \boxed{w} & \text{on } \Gamma_D \end{cases}$$

Neumann boundary condition (b.c.)

control

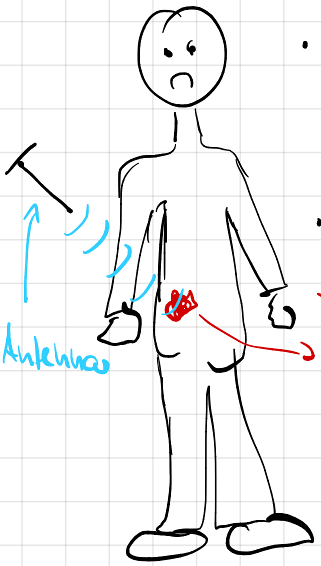
Dirichlet b.c.

and subject to the box control constants

$$0 \leq w(x) \leq w_{max} \qquad x \in \Gamma_D.$$

This is an example of linear-quadratic elliptic control problem with boundary control (and possibly box control constraints).

Example 3: Regional hyperthermia as cancer therapy



- Idea is to heat up the cancer tumor through radio/microwaves emitted by some antenna.
- Goal is to reach a therapeutic target temperature $z_d \approx 43°C$ without damaging healthy tissue in Ω_{hea} by "overcooking" it, so temperature y should be below some limit temperature z_{lim}.

Usual therapy time is 40 - 60 min.

Taken from "Mathematical cancer therapy planning in deep regional hyperthermia" by Deuflhard, Schiela, Weiser, Acta Numerica (2012)

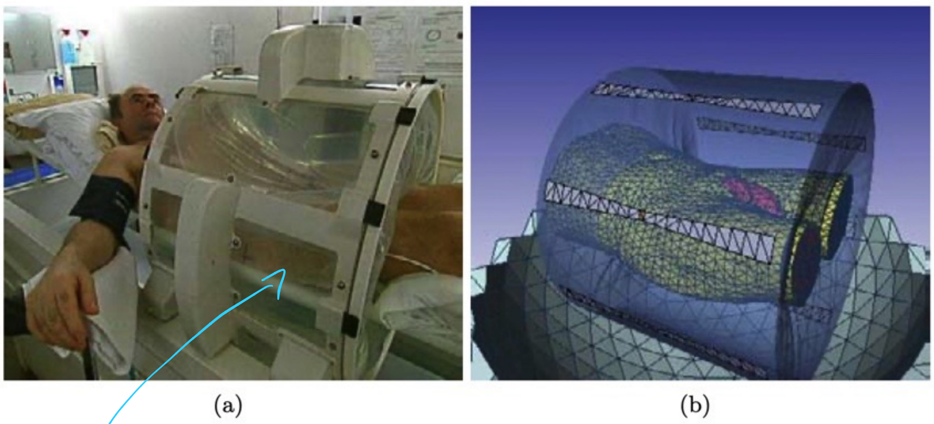


Figure 2.1. Hyperthermia treatment planning. (a) Real patient in hospital.[1] (b) Virtual patient in the virtual lab.

A simplified formulation as optimal control problem could be

$$J(u,y) = \underbrace{\frac{1}{2}\int_{\Omega_{tum}} |z_d - y(u)|^2}_{a)} + \underbrace{\frac{\alpha}{2}\int_{\Omega_{hea}} (y(u) - z_{lim})_+^2}_{b)} \quad \text{subject to}$$

only penalize if $y(u) > z_{lim}$

$$\begin{cases} \rho c \dfrac{\partial y}{\partial t} = \nabla \cdot (\kappa \nabla y) - c_b w (y - y_{blood}) + u & \text{in } \Omega \\ -\kappa \dfrac{\partial y}{\partial n} = h(y - y_{bolus}) & \text{on } \partial\Omega \end{cases}$$

Bio-heat equation

Robin boundary condition

- Alternative to avoid hot spikes in healthy tissue: Remove b) from J, instead required that $y(x) \le y_{lim}$ $x \in \Omega_{healt}$.

This is an example of a linear-quadratic parabolic control problem, possibly with state inequality constraints.