Project. Piecewise linear solutions to the Hunter–saxton equation
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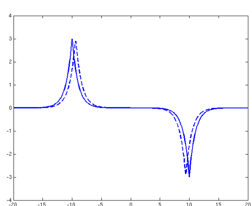
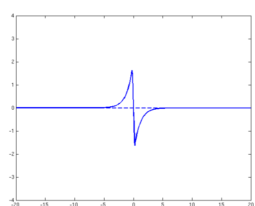
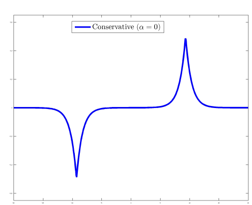
Background. The Hunter–Saxton equation, given by [1]

$$u_t + uu_x = \frac{1}{4}\left(\int_{-\infty}^{x} u_x^2(t,y)dy - \int_{x}^{\infty} u_x^2(t,y)dy\right)$$

serves as a model for waves in the director field of nematic liquid crystals, which one finds in various touch screens. From a pure mathematical point of view this equation attracted considerable attention since a huge class of solutions enjoys wave breaking within finite time. By this we mean that there are a lot of solutions which describe the same phenomenon that you can observe when a wave breaks near a shore. A huge amount of energy accumulates in one point, while the wave tips over. This means roughly that $u_x(t,x)$, the spatial derivative of the wave profile $u(t,x)$, no longer exists for all points (t,x). At the same time the function describing the energy of $u(t,x)$, turns into a delta function. Dependent on how the concentrated energy is manipulated one obtains different solution concepts. Keeping the energy in the system yields the so-called conservative solutions. If you think, for example of a wave profile consisting of one peak traveling to the right and a second one to the left like in the pictures below, then the solution looks

like the zero solution when wave breaking occurs. By keeping the energy unchanged, one obtains that the two peaks pass through each other. If we had chosen to continue the solution by being equal to zero, we would have dissipated all the accumulated energy.

Problem. Rather recently a generalization has been introduced, the so-called two-component Hunter-Saxton system,

$$u_t(t,x) + uu_x(t,x) = \frac{1}{4}\left(\int_{-\infty}^{x} (u_x^2 + \varepsilon\rho^2)(t,y)dy - \int_{x}^{\infty} (u_x^2 + \varepsilon\rho^2)(t,y)dy\right), \tag{0.1a}$$

$$\rho_t + (u\rho)_x = 0 \tag{0.1b}$$

[1]Here u_x and u_t denote the partial derivatives with respect to x and t, respectively.

where $\varepsilon > 0$. It has been shown that if there are no intervals where both $u_0(x)$ is decreasing and $\rho_0(x) = 0$, no wave breaking occurs. However, as $\varepsilon \to 0$ it seems like the first equation in the two-component Hunter–Saxton system reduces to the Hunter–Saxton equation. Hence a current question of interest is to understand the limiting process. As a starting point we propose the following student project.

1. Let

$$u_0(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 \leq x \leq 1 \\ 1, & 1 \leq x \end{cases} \qquad \rho_0(x) = \begin{cases} 0, & x \leq 0, \\ 1, & 0 \leq x \leq 1 \\ 0, & 1 \leq x. \end{cases} \tag{0.2}$$

Compute the unique solution $u_\varepsilon(t, x)$ for $\varepsilon > 0$ to (0.1). Afterwards show that $u(t, x) = \lim_{\varepsilon \to 0} u_\varepsilon(t, x)$ coincides with the energy preserving solution of the HS equation. What can you say about the limiting energy function?

2. Let

$$u_0(x) = \begin{cases} 1, & x \leq 0 \\ 1 - x, & 0 \leq x \leq 1 \\ 0, & 1 \leq x \end{cases} \qquad \rho_0(x) = \begin{cases} 0, & x \leq 0, \\ 1, & 0 \leq x \leq 1 \\ 0, & 1 \leq x. \end{cases} \tag{0.3}$$

Compute the unique solution $u_\varepsilon(t, x)$ for $\varepsilon > 0$ to (0.1). Afterwards show that $u(t, x) = \lim_{\varepsilon \to 0} u_\varepsilon(t, x)$ coincides with the energy preserving distributional solution of the HS equation. What can you say about the limiting energy function?

Requirements. Matematikk 1, 2 and 4, or equivalent courses. If you have never stumbled into generalized functions or distributions before, that is no problem.

Workload. About 80 hours.

References

[1] A. Bressan, H. Holden and X. Raynaud. Lipschitz metric for the Hunter–Saxton equation. J. Math. Pures Appl. 94:68–92, 2010.

[2] A. Nordli. On the Hunter–Saxton equation. (master thesis). Master thesis, NTNU, 2012.

[3] A. Nordli. A Lipschitz metric for conservative solutions of the two-component Hunter–Saxton system. arXiv:1502.07512.