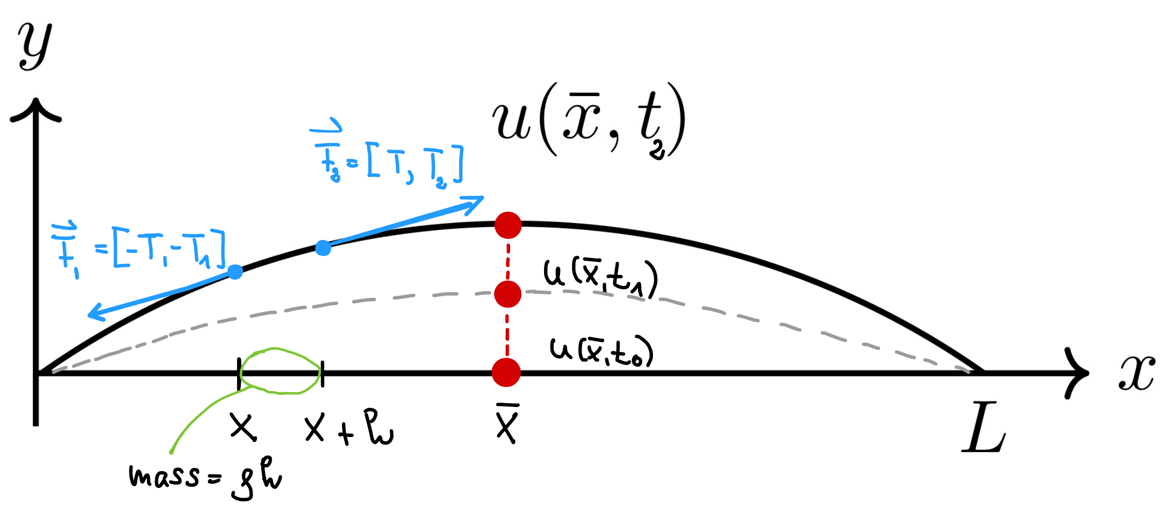
Lecture 13 and 14: The wave equation

1. Derivation of the wave equation

- Model problem: 1-dimensional vibrating string which is fixed at $x = 0$ and $x = L$.
- $u(x,t)$ describes
- Assume uniform mass density $\rho(x)$ [kg/m], no gravity forces ⇒ only forces acting are stretch forces / tension acting parallely to the string



- Since we assume that there is no horizontal movement of the point $\bar{x}$, the horizontal component of F_1 and F_2 must cancel out each other: $(\vec{F}_1)_x = T = -(\vec{F}_2)_x$.
- Using Newton's second law $\vec{F} = m\vec{a}$, the vertical acceleration is given by $h\rho\, \partial_t^2 u(x,t) = T_2 - T_1$, or

$$\rho\, \partial_t^2 \partial_t^2 u(x,t) = \frac{T_2 - T_1}{h}$$

$$\frac{\rho}{T}\, \partial_t^2 u^2(x,t) = \frac{\frac{T_2}{T} - \frac{T_1}{T}}{h} = \frac{\partial_x u(x+h,t) - \partial_x u(x,t)}{h} \xrightarrow{h \to 0} \partial_x^2 u(x,t).$$

Setting $c^2 = \frac{T}{\rho}$ we obtain the wave equation :

$$\partial_t^2 u(x,t) = c^2 \, \partial_x^2 u(x,t) \qquad x \in (0,L), \; t > 0.$$

+ Dirichlet b.c. $u(0,t) = u(L,t) = 0$.

- We need also some initial condition:

i.c. I $\qquad u(x,0) = f(x) \qquad$ (initial displacement).

i.c. II $\qquad \partial_t u(x,0) = g(x) \qquad$ (initial velocity) .

2. Solution of the wave equation on an interval

Subtitle: Separation of variables again

- Consider

$$\boxed{\text{I}}\quad \begin{cases} \text{PDE} : & \partial_t^2 u(x,t) = c^2 \partial_x^2 u(x,t) & x \in (0,L),\ t>0 \\ \text{Dirichlet b.c.} : & u(0,t) = u(L,t) = 0 & t>0 \\ \text{2 Initial conditions} \quad \text{i.c. I}: & u(x,0) = f(x) & x \in [0,L] \\ \text{i.c. II}: & \partial_t u(x,0) = g(x) & x \in [0,L] \end{cases}$$

See lecture 11.

- We proceed similar as for the heat equation and try to construct a solution via the method of separation of variables:
- Ansatz: $u(x,t) = F(x)\,G(t)$
- Plugging this into the PDE yields

$$G''(t)\,F(x) = c^2 G(t)\,F''(x) \Rightarrow \frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} \overset{!}{=} -k$$

for some so far unknown constant k since the variables x and t vary independent of each other. Thus we again left with

1) $$F''(x) + k\,F(x) = 0$$

2) $$G''(t) + k c^2 G(t) = 0$$

Starting with 1) the exact considerations as in lecture 11 we know that the general solution for 1) can we written as

$$F(x) = \begin{cases} \tilde{A}\, e^{\sqrt{-k}\,x} + \tilde{B}\, e^{-\sqrt{-k}\,x}, & k \neq 0 \\ \tilde{A} + \tilde{B}\,x & k = 0 \end{cases}$$

As discussed in lecture 11, we can rule out the boring / non-working case $\mathcal{E} = 0$ and $\mathcal{E} < 0$. For $\mathcal{E} > 0$ $\sqrt{-\mathcal{E}} = i\sqrt{\mathcal{E}}$ is imaginary and thus

$$F(x) = \tilde{A}\, e^{i\sqrt{\mathcal{E}}x} + \tilde{B}\, e^{-i\sqrt{\mathcal{E}}x}$$

$$= A\cos\sqrt{\mathcal{E}}x + B\sin\sqrt{\mathcal{E}}x$$

and taking Dirichlet b.c. into account led us to a nontrivial solution F if

$$\mathcal{E} = \left(\frac{n\pi}{L}\right)^2 \quad n \in \mathbb{N}, \quad \text{then}$$

$$F(x) = \sin\frac{n\pi}{L}x$$

solves 1) and respects our Dirichlet b.c.

- Turning to 2). For our particular choice of $\mathcal{E} = \left(\frac{n\pi}{L}\right)^2$, we have

$$G''(t) + \left(\frac{cn\pi}{L}\right)^2 G(t) = 0,$$

with the general solution

$$G_n(t) = A_n\cos\frac{cn\pi}{L}t + B_n\sin\frac{cn\pi}{L}t,$$

which in turn gives us a family of functions

$$u_n(x,t) = F(x)G_n(t)$$
$$= \left(A_n\cos\frac{cn\pi}{L}t + B_n\sin\frac{cn\pi}{L}t\right)\sin\frac{n\pi}{L}x$$

- We still need to incorporate the initial conditions!

- As before we superimpose our solution family

$$u(x,t) = \sum_{n=1} \left(A_n \cos \frac{cn\pi}{l} t + B_n \sin \frac{cn\pi}{l} t \right) \cdot \sin \frac{n\pi}{l} x$$

- Now try to incorporate i.c. I which yields for $t = 0$:

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{l} x$$

Thus, we need to develop (again !) $f(x)$ into a Fourier / sin series, and thus $\{A_n\}_{n=1}^{\infty}$ are the Fourier coefficients to the odd extension f_o of f:

$$\boxed{A_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi}{l} x \, dx.}$$

- Incorporating i.c. II gives

$$g(x) = \partial_t u(x,0) = \sum_{n=1} \left(-\frac{cn\pi}{l} \cdot A_n \cdot \sin \frac{cn\pi}{l} t + \frac{cn\pi}{l} \cdot B_n \cdot \cos \frac{cn\pi}{l} t \right) \Bigg|_{t=0} \cdot \sin \frac{n\pi}{l} x$$

$$= \sum_{n=1} \underbrace{\frac{cn\pi}{l} \cdot B_n}_{=: \tilde{B}_n} \sin \frac{n\pi}{l} x .$$

- So $\{\tilde{B}_n\}_{n=1}^{\infty}$ are the Fourier coefficients of the odd extension g_o; and thus

$$\boxed{B_n = \frac{2}{cn\pi} \int_0^l g(x) \sin \frac{n\pi}{l} x \, dx.}$$

Theorem 1

The wave equation of the form $\boxed{1}$ has the solution

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{cn\pi}{L} t + B_n \sin \frac{cn\pi}{L} t \right) \sin \frac{n\pi}{L} x$$

with

$$\boxed{\begin{aligned} A_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx, \\ B_n &= \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx. \end{aligned}}$$

Example: See Kreyszig, Section 12.3 Example 1:
"Vibrating String if the Initial Deflection is Triangular".

3. Solution for a flute

Problem we consider is now

$$\boxed{2}\begin{cases} \text{PDE} & \partial_t^2 u(x,t) = c^2 \partial_x^2 u(x,t),\ x \in (0,L),\ t > 0 \\ \text{Neumann b.c.} & \partial_x u(0,t) = \partial_x u(L,t) = 0 \\ \text{i.c. I} & u(x,0) = f(x) \\ \text{i.c. II} & \partial_t u(x,0) = g(x). \end{cases}$$

We won't solve this problem in detail, the idea is exactly the same as in Section 2, see also Jorten's L.N.

To work out the details starting from Jorten's L.N. and Section 2 will be part of Exercise set 7.

4. Wave equation on the entire x-Axis: D'Alembert

We want to solve

$$\begin{cases} \partial_t^2 u(x,t) = c^2 \partial_x u(x,t) & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x) & \text{i.c. I} \\ \partial_t u(x,0) = g(x) & \text{i.c. II} \end{cases}$$

- Ansatz: $u(x,t) = \phi(x+ct) + \psi(x-ct)$

for two functions $\phi : \mathbb{R} \to \mathbb{R}$, $\psi : \mathbb{R} \to \mathbb{R}$ which are twice differentiable. Then

$$\partial_t^2 u(x,t) = c^2 \phi''(x+ct) + c^2 \psi''(x-ct)$$

$$c^2 \partial_x^2 u(x,t) = c^2 \phi''(x+ct) + c^2 \psi''(x-ct)$$

- How to incorporate initial data?

I.C. I: $u(x,0) = \phi(x) + \psi(x) = f(x)$ 1)

I.C. I: $\partial_t u(x,0) = c\phi'(x) - c\psi'(x) = g(x)$

integrate

$$\Rightarrow \quad \phi(x) - \psi(x) = \frac{1}{c} \int^x g(x) + D. \qquad 2)$$

1) ⊕ 2) gives

$$2\phi(x) = f(x) + \frac{1}{c} \int^x g(x) + D$$

1) ⊖ 2) gives

$$2\psi(x) = f(x) - \frac{1}{c} \int^x g(x) - D$$

- If we insert this into our ansatz $u(x,t) = \phi(x+ct) + \psi(x-ct)$:

$$
\begin{aligned}
u(x,t) &= \phi(x+ct) + \psi(x-ct) \\
&= \frac{1}{2} f(x+ct) + \frac{1}{2c}\int^{x+ct} g + \frac{1}{2} D \\
&\quad + \frac{1}{2} f(x-ct) - \frac{1}{2c}\int^{x-ct} g - \frac{1}{2} D \\
&= \frac{1}{2}\big(f(x+ct) - f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy.
\end{aligned}
$$

- We summarizes this in the following

Theorem 2

The problem

$$
\begin{cases}
\partial_t^2 u(x,t) = c^2 \partial_x u(x,t) & x \in \mathbb{R},\ t > 0 \\
u(x,0) = f(x) & \text{i.c. I} \\
\partial_t u(x,0) = g(x) & \text{i.c. II}
\end{cases}
$$

has the solution

$$
\boxed{u(x,t) = \frac{1}{2}\big(f(x+ct) - f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy.}
$$