P1

Check axioms.

i) $\tilde{B}_0 = B_1 - B_1 = 0$

ii) $\Delta\tilde{B}_{t_k} = \tilde{B}_{t_{k+1}} - \tilde{B}_{t_k} = B_{t_{k+1}+1} - B_{t_k+1}$

$= \Delta B_{t_k+1} \sim N(0, t_{k+1} - t_k)$

iii) $0 \le t_1 < t_2 < \dots \le t_n$

$(\tilde{B}_{t_1}, \Delta\tilde{B}_{t_1}, \dots, \Delta\tilde{B}_{t_{n-1}})$

$= (B_{t_1+1}, \Delta B_{t_1+1}, \dots, \Delta B_{t_{n-1}+1})$

indep. R.V.'s.

P2

a) $\int_0^T \phi_N \, dB_s := \sum_{j=0}^{N} \phi_N(t_j, \omega) \Delta B_{t_j} = \sum_{j=0}^{N} B_{t_j}^2 \Delta B_{t_j}$

where $\Delta B_{t_j} = B_{t_{j+1}} - B_{t_j}$

$\int_0^T B_s^2 \, dB_s := \lim_{\substack{N \to \infty \\ \max|\Delta t_k| \to 0}} \int_0^t \phi_N \, dB_s$ in $L^2(\Omega)$

(Need $E \int_0^t |\phi_N - B_s^2|^2 ds \xrightarrow[\substack{N \to \infty \\ \max_k |\Delta t_k| \to 0}]{} 0$ which is ok here since B_t^2 is t-cont. and $\mathcal{F}_t$-adapted)

P2 b):

Compute exp and var of

(i) $X_t = \int_0^t s \, dB_s$

(ii) $Y = \int_0^T B_s^2 \, dB_s$

Ito-int. have exp. 0

$$\Longrightarrow \quad E(X_t) = 0 = E(Y)$$

Use Ito-isometry:

$$Var(X_t) = E|X_t|^2 = E\left| \int_0^t s \, dB_s \right|^2$$

$$\overset{\text{Ito-iso}}{=} \int_0^t E(s^2) \, ds = \underline{\underline{\frac{1}{3} t^3}}$$

$$Var(Y) = E|Y|^2 = E\left| \int_0^T B_s^2 \, dB_s \right|^2$$

$$\overset{\text{Ito-iso}}{=} \int_0^T \underbrace{E(B_s^4)}_{= 3s^2} \, ds = \underline{\underline{T^3}}$$

List of formulas

P2 c)

Let

$$\varphi = B_1^2(\omega)\chi_{[1,\infty)}(t)$$

Show by a direct argument that

$$X_t = \int_0^t \varphi(\omega,s)\,dB_s$$

is a mart. w.r.t. filtr. $\{\mathcal{F}_t\}_t$ of B.M. B_t.

$$X_t = \begin{cases} 0 & 0 \le t < 1 \\ B_1^2(\omega)(B_t - B_1) & t \ge 1 \end{cases}$$

i) X_t $\mathcal{F}_t$-meas $\left(\text{since } \{\emptyset, F\} \subset \mathcal{F}_s \subset \mathcal{F}_t, s < t\right)$

ii) $E|X_t| \le E\left(B_1^2 |B_t - B_1|\right)$

$$\overset{\text{indep.}}{\underset{\text{incr.}}{=}} E(B_1^2)\,E|B_t - B_1|$$

$$\underset{C-S}{\le} E(B_1^2)\sqrt{E|B_t - B_1|^2} = 1 \cdot t < \infty$$

iii) $E(X_t | \mathcal{F}_s) \overset{t>s}{=} E(X_s | \mathcal{F}_s) + E(X_t - X_s | \mathcal{F}_s)$

$$= X_s + E(X_t - X_s | \mathcal{F}_s)$$

since X_s $\mathcal{F}_s$-meas. (i))

4.)

$$X_t - X_s = \begin{cases} 0 & t < 1 \\ B_1^2 (B_t - B_1) & s < 1,\ t \geq 1 \\ B_1^2(\omega)(B_t - B_s) & \text{otherwise} \end{cases}$$

$$E(0 \mid \mathcal{F}_s) = 0$$

$$E(B_1^2 (B_t - B_1) \mid \mathcal{F}_s)$$

$$\overset{s<1}{\underset{\text{telescope}}{=}} E\Big(\underbrace{E(B_1^2 (B_t - B_1) \mid \mathcal{F}_1)}_{B_1^2 \underbrace{E(B_t - B_1 \mid \mathcal{F}_1)}_{= E(B_t - B_1) = 0}} \Big| \mathcal{F}_s \Big) = 0$$

$\mathcal{F}_1$-meas.

indep. of $\mathcal{F}_1$

$$E(B_1^2 (B_t - B_s) \mid \mathcal{F}_s)$$

$$\overset{B_1\ \mathcal{F}_s\text{-meas}}{\underset{s \geq 1}{=}} B_1^2 \underbrace{E(B_t - B_s \mid \mathcal{F}_s)}_{= E(B_t - B_s) = 0} = 0$$

indep. of $\mathcal{F}_s$

Hence X_t is $\mathcal{F}_t$ mart.

P3 a)

$$\ddot{X} + \dot{X} = W + \sin t$$

$$\Updownarrow$$

$$\begin{cases} \dot{X}_1 = X_2 \\ \dot{X}_2 = -X_2 + W + \sin t \end{cases}$$

$\downarrow$ Ito $\dot{X}_i = \frac{dX_i}{dt}$, " $W = \frac{dB_t}{dt}$ " , mult. by dt

$$\begin{cases} dX_1 = X_2\, dt \\ dX_2 = -X_2\, dt + dB_t + \sin t\, dt \end{cases}$$

or

$$\begin{cases} X_1(t) = x_1 + \int_0^t X_2(t)\, dt \\ X_2(t) = x_2 - \int_0^t X_2(t)\, dt + \int_0^t dB_s + \int_0^t \sin t\, dt \end{cases}$$

P3 b)

Brownian motion w. outside periodic forcing

$$m\ddot{X} + k\dot{X} = f_1 W + f_2 \sin t$$

$$dX_1 = X_2\,dt$$

$$dX_2 = -X_2\,dt + dB_t + \sin t$$

2nd eq'n uncoupled, solve it using int. factor.

Homogeneous eq'n,

$$dX_2 = -X_2\,dt,$$

has int. factor e^t:

$$d(e^t X_2) \overset{\text{Ito}}{=} e^t X_2\,dt + e^t dX_2 + 0$$
$$= 0$$

Let $f(x,t) = xe^t$. Then

$$f_t = xe^t, \quad f_x = e^t, \quad f_{xx} = 0$$

If X_2 sol'n (and hence Ito proc.), and since f is C^2, we may use Ito's formula:

$$d\,f(X_2,t) = f_t\,dt + f_x\,dX_2 + \tfrac{1}{2}\,\varepsilon^2 f_{xx}$$

$$= X_2 e^t\,dt + e^t\left(-X_2\,dt + dB_t + \sin t\,dt\right) + 0$$

$$= e^t\left(dB_t + \sin t\,dt\right)$$

$$\Rightarrow f(X_2(t),t) = f(X_2(0),0) + \int_0^t e^s\,dB_s + \int_0^t e^s \sin s\,ds$$

$$\Rightarrow e^t X_2(t) = x_2 + \int_0^t e^s\,dB_s + \int_0^t e^s \sin s\,ds$$

$$X_2(t) = e^{-t} x_2 + \int_0^t e^{-(t-s)}\,dB_s + \int_0^t e^{-(t-s)} \sin s\,ds$$

P4

X_t str. sol'n if

i) X_t Ito proc. $\begin{cases} X_t \ \mathcal{F}_t\text{-adap., jointly meas.} \\ \int_0^t |b(s,X_s)| + |\sigma(s,X_s)|^2 ds < \infty \text{ a.s. } \forall t > 0 \end{cases}$

ii) (T) holds a.s.

Sufficient cond'ns for ex. and uniqueness:

(A1) $|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq L|x-y|$

for all t, x, y. and

(A2) $Z \in L^2(\Omega)$, Z indep. of $\mathcal{F}_\infty = \mathcal{F}_{\{B_s : 0 \leq s < \infty\}}$

P5

The eq'n may be written as

$$(*)\quad \begin{cases} u_t = Au & , t>0 \\ u = f & , t=0 \end{cases}$$

where

$$A = \frac{1}{2}\partial_x^2 + \partial_x + y\,\partial_y$$

is the generator of the following Itô diffusion

$$\begin{cases} dX_t = dt + dB_t \\ dY_t = Y_t\,dt \end{cases} \Rightarrow \begin{cases} X_t^x = x + t + B_t \\ Y_t^y = y e^t \end{cases}$$

Since $f \in C_c^2$,
Eq'n $(*)$ is then the "backward" Kolmogorov eq'n satisfied by

$$u(t,x,y) = E\left(f(X_t^x, Y_t^y)\right)$$

$$= \int_{\mathbb{R}} f(x+t+z, ye^t)\frac{1}{\sqrt{2\pi t}}e^{-\frac{z^2}{2t}}\,dz.$$

where we used the prob. density func'n of $B_t \sim N(0,t)$ given in formulas at the end of the exam.

P6

The generator of X_t^x is

$$A = \frac{1}{2}(2+\sin x)^2 \partial_x^2$$

If $f \in C_c^2$ and τ st. t. w. $E(\tau) < \infty$,
then Dynkin's formula gives that

$$f(X_\tau^x) = f(x) + E\int_0^\tau Af(X_s^x)\,ds$$

Take $f_k \in C_c^2$ s.t. $f_k = x^2$ for $|x| \le R$

and let $\tau_k = \min(\tau, k)$.

Note that τ st. t. and $E(\tau) \le k$.

By Dynkin

$$|X_{\tau_k}^x|^2 = |x|^2 + E\int_0^{\tau_k} \underbrace{(2+\sin X_s^x)^2}_{\ge 1} ds$$

$$\ge |x|^2 + 1 \cdot E(\tau_k)$$

$$\Rightarrow E(\tau_k) \le |X_{\tau_k}^x| - |x|^2 \le R^2 - |x|^2 < \infty$$

By monotone conv. thm

$$E(\tau) = \lim E(\tau_k) \le R^2 - |x|^2 < \infty$$

P7

Let $f(x,y) = |x-y|^2$, then $f \in C^2$ and

$f_x = (x-y)$, $f_y = -(x-y)$, $f_{xx} = 1$

Then $f(X_t, X_t^\varepsilon)$ is an Ito proc. and by the Ito formula

$$df(X_t, X_t^\varepsilon) = f_x \, dX_t + f_y \, dX_t^\varepsilon + \tfrac{1}{2} f_{xx} \varepsilon^2 dt + \tfrac{1}{2} f_{yy} 0 \, dt + f_{xy} 0 \, dt$$

$$= (X_t - X_t^\varepsilon)(b(X_t)\,dt + \varepsilon \, dB_t) - (X_t - X_t^\varepsilon)\, b(X_t^\varepsilon) + \tfrac{1}{2}\varepsilon^2 dt$$

$$= (X_t - X_t^\varepsilon)(b(X_t) - b(X_t^\varepsilon))\,dt + \tfrac{1}{2}\varepsilon^2 dt + \varepsilon \, dB_t$$

Hence

$$|X_t - X_t^\varepsilon|^2 = \underbrace{|x_0 - x_0|^2}_{=0} + \int (\cdots)\,dt + \tfrac{1}{2}\varepsilon^2 t + \varepsilon B_t$$

$$\leq \int_0^t L |X_s - X_s^\varepsilon|^2 ds + \tfrac{1}{2}\varepsilon^2 t + \varepsilon B_t$$

since $(X_t - X_t^\varepsilon)(b(X_t) - b(X_t^\varepsilon)) \leq |X_t - X_t^\varepsilon| \cdot L |X_t - X_t^\varepsilon|$

Taking exp. and using Fubini,

$$\underbrace{E|X_t - X_t^\varepsilon|^2}_{v(t)} \leq L\int_0^t \underbrace{E|X_s - X_s^\varepsilon|^2}_{v(s)} ds + \frac{1}{2}\varepsilon^2 t$$

and then using Grönwall (list of formulas),

$$v(t) \leq \frac{1}{2}\varepsilon^2 t\, e^{Lt}$$

Alternative: Use SDE directly

$$\underbrace{E|X_t - X_t^\varepsilon|^2}_{v(t)} \overset{(2),(3)}{=} E\left|x_0 + \int_0^t b(X_s)ds - x_0 - \int_0^t b(X_s^\varepsilon)ds + \varepsilon B_t\right|^2$$

$(a+b)^2 \leq 2a^2 + 2b^2$

$$\leq 2E\left|\int_0^t \left(b(X_s) - b(X_s^\varepsilon)\right) ds\right|^2 + 2E|\varepsilon B_t|^2$$

$$\leq 2E\left|\int_0^t L|X_s - X_s^\varepsilon| ds\right|^2 + 2\varepsilon^2 t$$

Jensen

$$\leq 2Et\int_0^t L^2|X_s - X_s^\varepsilon|^2 ds + 2\varepsilon^2 t$$

$$= 2tL^2 \int_0^t v(s)\, ds + 2\varepsilon^2 t$$

Grönwall

$$\Longrightarrow \quad v(t) \leq 2\varepsilon^2 t\, e^{2L^2 t}$$