Problems 1–6, 7a), 7b), 8 and 9 all have equal weight.

Problem 1 For which values $a \in \mathbb{R}$ is the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2a & -4 \\ a & 1 & -a \end{bmatrix}$$

invertible?

Problem 2 Compute $\operatorname{Re} z$ and $\operatorname{Im} z$ when

$$z = \mathrm{i}^{\left(\sqrt{2}+\mathrm{i}\sqrt{2}\right)^4} - 3\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}.$$

Write the answer in the simplest possible way.

Problem 3 Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & -9 \\ 2 & -1 & 8 \end{bmatrix}.$$

- Obtain a basis for $\operatorname{Col} A$ and $\operatorname{Null} A$.
- Does the system

  $$A\boldsymbol{x} = \boldsymbol{b}$$

  have a solution for all $\boldsymbol{b} \in \mathbb{R}^3$? If it has a solution for a specific $\boldsymbol{b}$, is the solution unique? Justify your answer.

Problem 4 Solve the initial-value problem

$$y''(t) - 2y'(t) + 3y(t) = 9t, \qquad y(0) = -1, \quad y'(0) = \sqrt{2}.$$

Problem 5 Suppose that $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are linearly independent vectors in a vector space V. Show that the three vectors

$$\boldsymbol{u}+\boldsymbol{v}, \qquad \boldsymbol{u}+\boldsymbol{w}, \qquad \text{and} \qquad \boldsymbol{v}+\boldsymbol{w}$$

also are linearly independent.

Problem 6 Let $V \subseteq \mathbb{R}^3$ be the linear span $\mathrm{Sp}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$.

- Find an orthogonal basis for V.
- Compute the orthogonal projection of $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ onto V.

Problem 7 Let A be the matrix

$$\begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & -2 & -1 \end{bmatrix}.$$

a) • Find the eigenvalues of A and • compute A^{31}.

b) • Solve the initial-value problem

$$\boldsymbol{y}'(t) = A\boldsymbol{y}(t), \qquad \boldsymbol{y}(0) = \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}$$

and • determine $\lim_{t\to\infty} \boldsymbol{y}(t)$, if the limit exists.

Problem 8 Use the least-squares method to find the second-order polynomial

$$p(x) = ax^2 + bx + c$$

that minimises the distance to the data points

x	-1	0	1	2
y	1	-2	0	5

(That is, find a, b and c.)

Problem 9 Let $\mathcal{M}_{n\times n}(\mathbb{R})$ be the vector space of real-valued $n \times n$ matrices and let

$$\mathcal{S}_n = \left\{ A \in \mathcal{M}_{n\times n}(\mathbb{R}) \ : \ A^\top = A \right\} \qquad (A^\top \text{ is the transpose of } A).$$

- Give a basis for $\mathcal{S}_n$ when $n = 2$.
- Find $\dim \mathcal{S}_n$ for all $n \in \mathbb{N}$.