If two harmonic functions u and v satisfy the Cauchy–Riemann equations in a domain D, they are the real and imaginary parts of an analytic function f in D. Then v is said to be a **harmonic conjugate function** of u in D. (Of course, this has absolutely nothing to do with the use of "conjugate" for $\bar{z}$.)

EXAMPLE 4 How to Find a Harmonic Conjugate Function by the Cauchy–Riemann Equations

Verify that $u = x^2 - y^2 - y$ is harmonic in the whole complex plane and find a harmonic conjugate function v of u.

Solution. $\nabla^2 u = 0$ by direct calculation. Now $u_x = 2x$ and $u_y = -2y - 1$. Hence because of the Cauchy–Riemann equations a conjugate v of u must satisfy

$$v_y = u_x = 2x, \qquad v_x = -u_y = 2y + 1.$$

Integrating the first equation with respect to y and differentiating the result with respect to x, we obtain

$$v = 2xy + h(x), \qquad v_x = 2y + \frac{dh}{dx}.$$

A comparison with the second equation shows that $dh/dx = 1$. This gives $h(x) = x + c$. Hence $v = 2xy + x + c$ (c any real constant) is the most general harmonic conjugate of the given u. The corresponding analytic function is

$$f(z) = u + iv = x^2 - y^2 - y + i(2xy + x + c) = z^2 + iz + ic. \qquad \blacksquare$$

Example 4 illustrates that *a conjugate of a given harmonic function is uniquely determined up to an arbitrary real additive constant.*

The Cauchy–Riemann equations are the most important equations in this chapter. Their relation to Laplace's equation opens a wide range of engineering and physical applications, as shown in Chap. 18.

PROBLEM SET 13.4

1. **Cauchy–Riemann equations in polar form.** Derive (7) from (1).

2–11 CAUCHY–RIEMANN EQUATIONS

Are the following functions analytic? Use (1) or (7).

2. $f(z) = iz\bar{z}$
3. $f(z) = e^{-x}\cos(y) - ie^{-x}\sin(y)$
4. $f(z) = e^x(\cos y - i\sin y)$
5. $f(z) = \text{Re}(z^2) - i\,\text{Im}(z^2)$
6. $f(z) = 1/(z - z^5)$
7. $f(z) = -i/z^4$
8. $f(z) = \text{Arg } z$
9. $f(z) = 3\pi^2/(z^3 + 4\pi^2 z)$
10. $f(z) = \ln|z| + i\,\text{Arg } z$
11. $f(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$

12–19 HARMONIC FUNCTIONS

Are the following functions harmonic? If your answer is yes, find a corresponding analytic function $f(z) = u(x, y) + iv(x, y)$.

12. $u = x^3 + y^3$
13. $u = -2xy$
14. $v = xy$
15. $u = -\dfrac{x}{x^2 + y^2}$
16. $u = \sin x \cosh y$
17. $v = (2x - 1)y$
18. $u = x^3 - 3xy^2$
19. $v = e^{-x}\sin 2y$

20. **Laplace's equation.** Give the details of the derivative of (9).

21–24 Determine a and b so that the given function is harmonic and find a harmonic conjugate.

21. $u = e^{-\pi x}\cos ay$
22. $u = \cos ax \cosh 2y$
23. $u = ax^3 + bxy$
24. $u = \cosh ax \cos y$

25. **CAS PROJECT. Equipotential Lines.** Write a program for graphing equipotential lines $u = \text{const}$ of a harmonic function u and of its conjugate v on the same axes. Apply the program to **(a)** $u = x^2 - y^2$, $v = 2xy$, **(b)** $u = x^3 - 3xy^2$, $v = 3x^2y - y^3$.

26. Apply the program in Prob. 25 to $u = e^x \cos y$, $v = e^x \sin y$ and to an example of your own.

27. Harmonic conjugate. Show that if u is harmonic and v is a harmonic conjugate of u, then u is a harmonic conjugate of $-v$.

28. Illustrate Prob. 27 by an example.

29. Two further formulas for the derivative. Formulas (4), (5), and (11) (below) are needed from time to time. Derive

(11) $\quad f'(z) = u_x - iu_y, \qquad f'(z) = v_y + iv_x.$

30. TEAM PROJECT. Conditions for $f(z) = $ const. Let $f(z)$ be analytic. Prove that each of the following conditions is sufficient for $f(z) = $ const.

(a) $\operatorname{Re} f(z) = $ const

(b) $\operatorname{Im} f(z) = $ const

(c) $f'(z) = 0$

(d) $|f(z)| = $ const (see Example 3)

13.5 Exponential Function

In the remaining sections of this chapter we discuss the basic elementary complex functions, the exponential function, trigonometric functions, logarithm, and so on. They will be counterparts to the familiar functions of calculus, to which they reduce when $z = x$ is real. They are indispensable throughout applications, and some of them have interesting properties not shared by their real counterparts.

We begin with one of the most important analytic functions, the complex **exponential function**

$$e^z, \qquad \text{also written} \qquad \exp z.$$

The definition of e^z in terms of the real functions e^x, $\cos y$, and $\sin y$ is

(1) $$e^z = e^x(\cos y + i \sin y).$$

This definition is motivated by the fact the e^z ***extends*** the real exponential function e^x of calculus in a natural fashion. Namely:

(A) $e^z = e^x$ for real $z = x$ because $\cos y = 1$ and $\sin y = 0$ when $y = 0$.

(B) e^z is analytic for all z. (Proved in Example 2 of Sec. 13.4.)

(C) The derivative of e^z is e^z, that is,

(2) $$(e^z)' = e^z.$$

This follows from (4) in Sec. 13.4,

$$(e^z)' = (e^x \cos y)_x + i(e^x \sin y)_x = e^x \cos y + ie^x \sin y = e^z.$$

REMARK. This definition provides for a relatively simple discussion. We could define e^z by the familiar series $1 + x + x^2/2! + x^3/3! + \cdots$ with x replaced by z, but we would then have to discuss complex series at this very early stage. (We will show the connection in Sec. 15.4.)

Further Properties. A function $f(z)$ that is analytic for all z is called an **entire function**. Thus, e^z is entire. Just as in calculus the ***functional relation***

(3) $$e^{z_1+z_2} = e^{z_1}e^{z_2}$$

Periodicity of e^x with period $2\pi i$,

$$e^{z+2\pi i} = e^z \quad \text{for all } z \tag{12}$$

is a basic property that follows from (1) and the periodicity of cos y and sin y. Hence all the values that $w = e^z$ can assume are already assumed in the horizontal strip of width 2π

$$-\pi < y \leqq \pi \qquad \text{(Fig. 336).} \tag{13}$$

This infinite strip is called a **fundamental region** of e^z.

EXAMPLE 1 Function Values. Solution of Equations

Computation of values from (1) provides no problem. For instance,

$$e^{1.4-0.6i} = e^{1.4}(\cos 0.6 - i \sin 0.6) = 4.055(0.8253 - 0.5646i) = 3.347 - 2.289i$$

$$|e^{1.4-1.6i}| = e^{1.4} = 4.055, \qquad \text{Arg } e^{1.4-0.6i} = -0.6.$$

To illustrate (3), take the product of

$$e^{2+i} = e^2(\cos 1 + i \sin 1) \qquad \text{and} \qquad e^{4-i} = e^4(\cos 1 - i \sin 1)$$

and verify that it equals $e^2e^4(\cos^2 1 + \sin^2 1) = e^6 = e^{(2+i)+(4-i)}$.

To solve the equation $e^z = 3 + 4i$, note first that $|e^z| = e^x = 5, x = \ln 5 = 1.609$ is the real part of all solutions. Now, since $e^x = 5$,

$$e^x \cos y = 3, \qquad e^x \sin y = 4, \qquad \cos y = 0.6, \qquad \sin y = 0.8, \qquad y = 0.927.$$

Ans. $z = 1.609 + 0.927i \pm 2n\pi i$ $(n = 0, 1, 2, \cdots)$. These are infinitely many solutions (due to the periodicity of e^z). They lie on the vertical line $x = 1.609$ at a distance 2π from their neighbors. ∎

To summarize: many properties of $e^z = \exp z$ parallel those of e^x; an exception is the periodicity of e^z with $2\pi i$, which suggested the concept of a fundamental region. Keep in mind that e^z is an *entire function.* (Do you still remember what that means?)

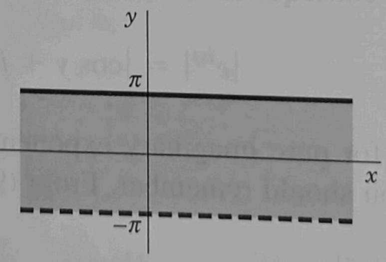

Fig. 336. Fundamental region of the exponential function e^z in the z-plane

PROBLEM SET 13.5

1. **e^z is entire.** Prove this.

2–7 Function Values. Find e^z in the form $u + iv$ and $|e^z|$ if z equals

2. $3 + 4i$
3. $2\pi i(1 - i)$
4. $0.6 - 1.8i$
5. $1 - 3\pi i$
6. $11\pi i/2$
7. $\sqrt{3} - \frac{\pi}{2}i$

8–13 Polar Form. Write in exponential form (6):

8. $\sqrt[n]{z}$
9. $3 - 4i$
10. $\sqrt{i}, \quad \sqrt{-i}$
11. $-\frac{3}{2}$
12. $1/(1 - z)$
13. $1 - i$

14–17 Real and Imaginary Parts. Find Re and Im of

14. $e^{-\pi z}$
15. $\exp(-z^2)$

16. $e^{1/z}$ **17.** $\exp(z^3)$

18. TEAM PROJECT. Further Properties of the Exponential Function. (a) Analyticity. Show that e^z is entire. What about $e^{1/z}$? $e^{\bar{z}}$? $e^x(\cos ky + i \sin ky)$? (Use the Cauchy–Riemann equations.)

(b) Special values. Find all z such that (i) e^z is real, (ii) $|e^{-z}| < 1$, (iii) $e^{\bar{z}} = \overline{e^z}$.

(c) Harmonic function. Show that $u = e^{xy} \cos(x^2/2 - y^2/2)$ is harmonic and find a conjugate.

(d) Uniqueness. It is interesting that $f(z) = e^z$ is uniquely determined by the two properties $f(x + i0) = e^x$ and $f'(z) = f(z)$, where f is assumed to be entire. Prove this using the Cauchy–Riemann equations.

19–22 Equations. Find all solutions and graph some of them in the complex plane.

19. $e^z = 1$ **20.** $e^z = 4 + 3i$

21. $e^z = 0$ **22.** $e^z = -2$

13.6 Trigonometric and Hyperbolic Functions. Euler's Formula

Just as we extended the real e^x to the complex e^z in Sec. 13.5, we now want to extend the familiar *real* trigonometric functions to *complex trigonometric functions*. We can do this by the use of the Euler formulas (Sec. 13.5)

$$e^{ix} = \cos x + i \sin x, \qquad e^{-ix} = \cos x - i \sin x.$$

By addition and subtraction we obtain for the *real* cosine and sine

$$\cos x = \tfrac{1}{2}(e^{ix} + e^{-ix}), \qquad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}).$$

This suggests the following definitions for complex values $z = x + iy$:

$$\text{(1)} \qquad \cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}), \qquad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

It is quite remarkable that here in complex, functions come together that are unrelated in real. This is not an isolated incident but is typical of the general situation and shows the advantage of working in complex.

Furthermore, as in calculus we define

$$\text{(2)} \qquad \tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z}$$

and

$$\text{(3)} \qquad \sec z = \frac{1}{\cos z}, \qquad \csc z = \frac{1}{\sin z}.$$

Since e^z is entire, $\cos z$ and $\sin z$ are entire functions. $\tan z$ and $\sec z$ are not entire; they are analytic except at the points where $\cos z$ is zero; and $\cot z$ and $\csc z$ are analytic except

PROBLEM SET 13.6

1–4 FORMULAS FOR HYPERBOLIC FUNCTIONS

Show that

1. $\cosh z = \cosh x \cos y + i \sinh x \sin y$

   $\sinh z = \sinh x \cos y + i \cosh x \sin y.$

2. $\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

   $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2.$

3. $\cosh^2 z - \sinh^2 z = 1, \quad \cosh^2 z + \sinh^2 z = \cosh 2z$

4. **Entire Functions.** Prove that $\cos z$, $\sin z$, $\cosh z$, and $\sinh z$ are entire.

5. **Harmonic Functions.** Verify by differentiation that $\operatorname{Im} \cos z$ and $\operatorname{Re} \sin z$ are harmonic.

6–12 Function Values. Find, in the form $u + iv$,

6. $\sin \frac{\pi}{2} i$

7. $\cos(-i), \quad \sin(-i)$

8. $\cos \pi i, \quad \cosh \pi i$

9. $\cosh(-2 + i), \quad \cos(-1 - 2i)$

10. $\sinh(3 + 4i), \quad \cosh(3 + 4i)$

11. $\sin \frac{\pi}{4} i, \quad \cos\left(\frac{\pi}{2} - \frac{\pi}{4} i\right)$

12. $\cos \frac{1}{2}\pi i, \quad \cos\left[\frac{1}{2}\pi(1 + i)\right]$

13–15 Equations and Inequalities. Using the definitions, prove:

13. $\cos z$ is even, $\cos(-z) = \cos z$, and $\sin z$ is odd, $\sin(-z) = -\sin z$.

14. $|\sinh y| \leqq |\cos z| \leqq \cosh y, |\sinh y| \leqq |\sin z| \leqq \cosh y.$ Conclude that the complex cosine and sine are not bounded in the whole complex plane.

15. $\sin z_1 \cos z_2 = \frac{1}{2}[\sin(z_1 + z_2) + \sin(z_1 - z_2)]$

16–19 Equations. Find all solutions.

16. $\sin z = 100$

17. $\cosh 2z = 0$

18. $\cosh z = -1$

19. $\sinh z = 0$

20. **Re tan z and Im tan z.** Show that

$$\operatorname{Re} \tan z = \frac{\sin x \cos x}{\cos^2 x + \sinh^2 y},$$

$$\operatorname{Im} \tan z = \frac{\sinh y \cosh y}{\cos^2 x + \sinh^2 y}.$$

13.7 Logarithm. General Power. Principal Value

We finally introduce the *complex logarithm*, which is more complicated than the real logarithm (which it includes as a special case) and historically puzzled mathematicians for some time (so if you first get puzzled—which need not happen!—be patient and work through this section with extra care).

The **natural logarithm** of $z = x + iy$ is denoted by $\ln z$ (sometimes also by $\log z$) and is defined as the inverse of the exponential function; that is, $w = \ln z$ is defined for $z \neq 0$ by the relation

$$e^w = z.$$

(Note that $z = 0$ is impossible, since $e^w \neq 0$ for all w; see Sec. 13.5.) If we set $w = u + iv$ and $z = re^{i\theta}$, this becomes

$$e^w = e^{u+iv} = re^{i\theta}.$$

Now, from Sec. 13.5, we know that e^{u+iv} has the absolute value e^u and the argument v. These must be equal to the absolute value and argument on the right:

$$e^u = r, \qquad v = \theta.$$

It is a ***convention*** that for real positive $z = x$ the expression z^c means $e^{c \ln z}$ where $\ln x$ is the elementary real natural logarithm (that is, the principal value Ln z ($z = x > 0$) in the sense of our definition). Also, if $z = e$, the base of the natural logarithm, $z^c = e^c$ is *conventionally* regarded as the unique value obtained from (1) in Sec. 13.5.

From (7) we see that for any complex number a,

$$a^z = e^{z \ln a}. \tag{8}$$

We have now introduced the complex functions needed in practical work, some of them (e^z, cos z, sin z, cosh z, sinh z) entire (Sec. 13.5), some of them (tan z, cot z, tanh z, coth z) analytic except at certain points, and one of them (ln z) splitting up into infinitely many functions, each analytic except at 0 and on the negative real axis.

For the **inverse trigonometric** and **hyperbolic functions** see the problem set.

PROBLEM SET 13.7

1–4 VERIFICATIONS IN THE TEXT

1. Verify the computations in Example 1.
2. Verify (5) for $z_1 = -i$ and $z_2 = -1$.
3. Prove analyticity of Ln z by means of the Cauchy–Riemann equations in polar form (Sec. 13.4).
4. Prove (4a) and (4b).

COMPLEX NATURAL LOGARITHM *ln z*

5–11 Principal Value Ln *z*. Find Ln z when z equals

5. -7
6. $8 + 8i$
7. $8 - 8i$
8. $1 \pm i$
9. $0.6 - 0.8i$
10. $-15 \pm 0.1i$
11. $-ei^2$

12–16 All Values of ln *z*. Find all values and graph some of them in the complex plane.

12. $\ln e$
13. $\ln 1$
14. $\ln(-5)$
15. $\ln(e^i)$
16. $\ln(4 - 3i)$
17. Show that the set of values of $\ln(i^2)$ differs from the set of values of $2 \ln i$.

18–21 Equations. Solve for z.

18. $\ln z = \pi i/2$
19. $\ln z = 4 - 3i$
20. $\ln z = e + \pi i$
21. $\ln z = 0.4 + 0.2i$

22–28 General Powers. Find the principal value. Show details.

22. $(2i)^{2i}$
23. $(1 + i)^{1-i}$
24. $(1 - i)^{1+i}$
25. $(-3)^{3-i}$
26. $(i)^{i/2}$
27. $(-1)^{2-i}$
28. $(3 + 4i)^{1/3}$
29. How can you find the answer to Prob. 24 from the answer to Prob. 23?
30. **TEAM PROJECT. Inverse Trigonometric and Hyperbolic Functions.** By definition, the **inverse sine** $w = \arcsin z$ is the relation such that $\sin w = z$. The **inverse cosine** $w = \arccos z$ is the relation such that $\cos w = z$. The **inverse tangent, inverse cotangent, inverse hyperbolic sine,** etc., are defined and denoted in a similar fashion. (Note that all these relations are ***multivalued***.) Using $\sin w = (e^{iw} - e^{-iw})/(2i)$ and similar representations of cos w, etc., show that

    (a) $\arccos z = -i \ln(z + \sqrt{z^2 - 1})$

    (b) $\arcsin z = -i \ln(iz + \sqrt{1 - z^2})$

    (c) $\operatorname{arccosh} z = \ln(z + \sqrt{z^2 - 1})$

    (d) $\operatorname{arcsinh} z = \ln(z + \sqrt{z^2 + 1})$

    (e) $\arctan z = \dfrac{i}{2} \ln \dfrac{i + z}{i - z}$

    (f) $\operatorname{arctanh} z = \dfrac{1}{2} \ln \dfrac{1 + z}{1 - z}$

    (g) Show that $w = \arcsin z$ is infinitely many-valued, and if w_1 is one of these values, the others are of the form $w_1 \pm 2n\pi$ and $\pi - w_1 \pm 2n\pi$, $n = 0, 1, \cdots$. (The *principal value of* $w = u + iv = \arcsin z$ is defined to be the value for which $-\pi/2 \leqq u \leqq \pi/2$ if $v \geqq 0$ and $-\pi/2 < u < \pi/2$ if $v < 0$.)