NTNU
Norwegian University of Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4285 Time Series**

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Examination date: November 25, 2020

Examination time (from–to): 09:00 – 13:00

Permitted examination support material: A: All printed and hand-written support material is allowed. All calculators are allowed.

Other information:

You may write in English or Norwegian.

All answers have to be justified. They should include enough details in order to see how they have been obtained.

All 25 sub-problems carry the same weight for grading.

Language: English

Number of pages: 4

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
sort/hvit ☐	**farger** ☒
skal ha flervalgskjema ☐	

Date Signature

Problem 1 Let $X = \{X_t \,|\, t \in \mathbb{Z}\}$ with

$$X = (1 + aB)(1 + bB^7)Z \tag{1}$$

where B is the backshift operator and $Z = \{Z_t \,|\, t \in \mathbb{Z}\} \sim \mathrm{WN}(0, \sigma^2)$. For calculations assume $a = -0.5282$, $b = -0.4920$, and $\sigma^2 = 0.4090$.

a) Show that $X \sim \mathrm{ARMA}(p, q)$ and determine p and q.

b) Prove that X is causal.

c) Is Z uniquely determined by X?

d) Calculate the covariance function γ of X.

e) How is the best linear predictor $\hat{X}_n$ of X_n given $X_1, \ldots, X_{256}$ defined, and is it unique?

Assume that

$$(1 - B)(1 - B^7)T = X \tag{2}$$

f) Is T weakly stationary?

g) Is T uniquely determined by Z?

h) Let $\hat{T}_n$ be the best linear predictor of T_n given $T_{-7}, \ldots, T_{256}$. Let $t_{-7}, \ldots, t_{256}$ be given. Demonstrate how $\hat{t}_{200}$, $\hat{t}_{257}$, and $\hat{t}_{300}$ can be calculated together with an uncertainty estimate.

i) State the necessary assumptions for the calculation of $\hat{t}_{200}$, $\hat{t}_{257}$, and $\hat{t}_{300}$.

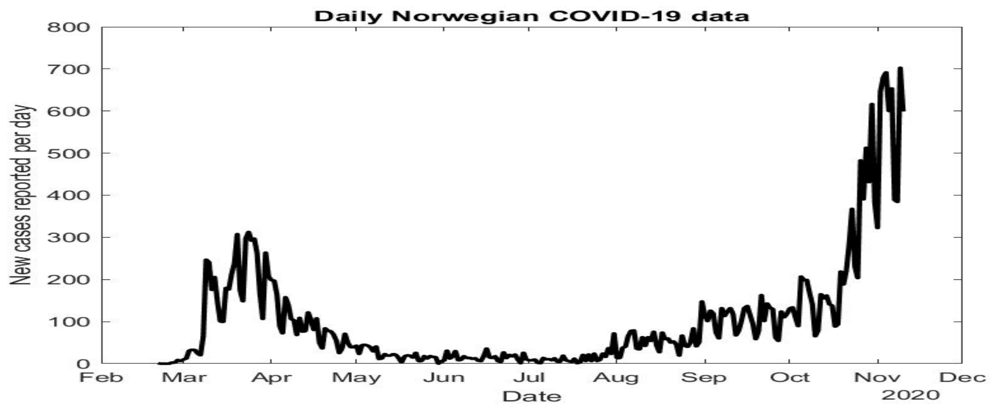

Figure 1: Daily numbers $y_{-7}, y_{-6}, \ldots, y_{256}$ of new reported cases of COVID-19 in Norway from February 21th to November 10th 2020.

Problem 2 The key assumption in the following is that the daily COVID-19 data in Figure 1 are realized values of parts of an integer valued stochastic process $Y = \{Y_t \,|\, t \in \mathbb{Z}\}$.

a) Explain why the key assumption is reasonable.

b) What are the possible values for Y_{300}?

c) Is it reasonable to assume that Y_{300} has a Poisson distribution?

d) Define mathematically the laws of Y and $(Y_{-7}, Y_{-6}, \ldots, Y_{256})$ in terms of the underlying probability space $(\Omega, \mathcal{E}, \mathrm{P})$.

e) How can $g(y) = \mathrm{P}(Y_{300} = y \,|\, Y_{-7} = y_{-7}, Y_{-6} = y_{-6}, \ldots, Y_{256} = y_{256})$ be calculated if the law of Y is known.

f) Assume that the law of Y is known and hence that g is known. Provide at least two reasonable predictions of y_{300} and explain how you can quantify the uncertainty.

g) Prove that

$$T = \log(\max(Y, 0.1)) \tag{3}$$

is a stochastic process.

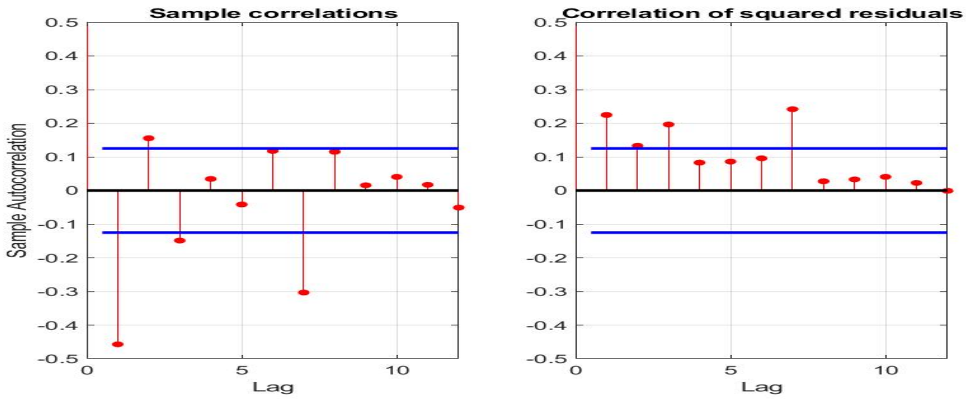

Figure 2: Correlation function estimates.

Problem 3 Let T be given by equation (3) and let

$$X = (1 - B)(1 - B^7)T \tag{4}$$

where B is the backshift operator. The left of Figure 2 shows the sample correlation $\hat{\rho}$ of $x_1, \ldots, x_{256}$ obtained from equations (4)-(3) and the data in Figure 1.

a) How is $\hat{\rho}$ calculated?

b) Figure 1 and Figure 2 can be used to motivate using the model

$$X = (1 + aB)(1 + bB^7)Z \tag{5}$$

where $Z \sim \text{WN}(0, \sigma^2)$. Why?

c) The empirical covariance function of $x_1, \ldots, x_{256}$ is $\hat{\gamma}(0) = 0.6252$, $\hat{\gamma}(1) = -0.2859$, ..., $\hat{\gamma}(7) = -0.1896$, ... Use this to estimate a, b and σ^2.

d) How can a, b, σ^2 be estimated by the maximum likelihood method?

e) How can the previous be used to forecast Y_{300}?

f) How can $z_1, \ldots, z_{256}$ be estimated using equation (5)?

g) Testing of residuals supports the assumption $Z \sim \text{WN}(0, \sigma^2)$. Is it reasonable to assume that Z is Gaussian?

h) The right part of Figure 2 motivates to consider a GARCH model for Z. Why?

i) Explain how a GARCH$(1, 1)$ model can be fitted to Z.