NTNU
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Department of Mathematical Sciences

Examination paper for **TMA4120 Mathematics 4K**

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Permitted examination support material: (Code C): Approved simple calculator.

Other information:
Every answer must be justified; describe clearly how you have reached your answers.

Language: English

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Informasjon om trykking av eksamensoppgave
Originalen er:
1-sidig ☐ **2-sidig** ☒
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skal ha flervalgskjema ☐

Date Signature

Problem 1 Find the solution $y(t)$ of the initial value problem

$$\begin{cases} \dfrac{d^3y}{dt^3} + 4\dfrac{dy}{dt} = \delta(t+100), \quad t > 0, \\ \\ y(0) = 0 = y'(0), \quad y''(0) = 5, \end{cases}$$

where $\delta(t)$ is the delta function.

Problem 2 Let the function $f(x)$ be defined on $[-1, 1]$ by

$$f(x) = \begin{cases} 1, & -1 \leq x \leq -\frac{1}{2}, \\ 0, & -\frac{1}{2} < x < \frac{1}{2}, \\ 1, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Show that the Fourier series of $f(x)$ is given by

$$S_f(x) = \frac{1}{2} - \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{\cos((2n+1)\pi x)}{2n+1}.$$

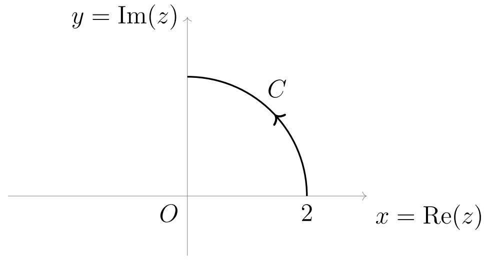
Sketch the sum $S_f(x)$ on the interval $[-2, 2]$ and determine the sum of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

Problem 3 Find all solutions $z \in \mathbb{C}$ of

$$\sin z = i.$$

Problem 4 Let C be the part of the circle $|z| = 2$ lying in the first quadrant oriented counter clockwisely.



Calculate the following integrals:

$$(i) \int_C z\,dz, \qquad (ii) \int_C \bar{z}\,dz, \qquad \text{and} \qquad (iii) \int_C \cosh(z)\,dz.$$

Problem 5 Let the function $f(z)$ be defined by

$$f(z) = \frac{1}{1-z} + \frac{1}{2-z} + \frac{1}{2022-z}.$$

Find all Taylor/Laurent series centered at $z_0 = 0$ that represent $f(z)$ for $|z| < 2$. Determine their regions of convergence.

Problem 6 The motion of damped soundwaves in an open-ended pipe can be modelled by the following (scaled) initial-boundary value problem for a damped wave equation on the interval $[0, 1]$,

(1) $u_{tt} + 2u_t = u_{xx}$, $x \in (0, 1)$, $t > 0$,

(2) $u_x(0, t) = 0 = u_x(1, t)$, $t > 0$,

(3) $u(x, 0) = 2 - \cos(4\pi x)$, $x \in [0, 1]$,

(4) $u_t(x, 0) = 0$, $x \in [0, 1]$.

a) Let $\tilde{u}(x, t) = X(x)T(t)$ be a solution of (1) and (2). (i) Find the differential equations and boundary conditions satisfied by X and T, (ii) check that

$$X_n(x) = \cos(n\pi x)$$

is a solution for $n = 0, 1, 2, 3, \ldots$, and (iii) find the solution T_n corresponding to X_n for $n = 0, 1, 2, 3, \ldots$.

Hint: The general solution of 2nd order differential equations can be found in the attachment.

b) Find the solution $u(x, t)$ of the problem (1), (2), (3), and (4).

Problem 7 Show that if $g(z)$ and $h(z)$ are analytic functions, $g(0) \neq 0 \neq h(0)$, and $h'(0) = 0$, then

$$\operatorname*{Res}_{z=0}\left[\frac{g(z)}{z^2 h(z)}\right] = \frac{g'(0)}{h(0)}.$$

Compute the integral

$$\oint_{|z-i|=2} \frac{e^{-i\pi z}}{z\sin(\pi z)}\, dz.$$

Hint: You can use without proof that $\operatorname{sinc}(z)$ is an analytic function such that $\operatorname{sinc}(0) = 1$, $\operatorname{sinc}'(0) = 0$, and $z\operatorname{sinc}(z) = \sin z$ for $z \in \mathbb{C}$.

Miscellaneous

- **Heaviside function** $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$, $u(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases}$
- **Dirac Delta function** $\delta(t-a)$ is zero except at $t = a$, $\int_{-\infty}^{\infty} \delta(t-a)dt = 1$, and $\int_{-\infty}^{\infty} g(t)\delta(t-a)dt = g(a)$ for any continuous function g.
- **Convolution**

  For functions defined on the real line:
  $$f * g\,(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy = \int_{-\infty}^{\infty} f(x-y)g(y)dy, \quad x \in \mathbb{R}.$$
  For functions defined only on the positive half-axis:
  $$f * g\,(x) = \int_0^x f(y)g(x-y)dy, \quad x > 0.$$

Laplace transform

- Definition: $\mathcal{L}[f](s) = F(s) = \int_0^{\infty} f(t)e^{-st}dt$

General formulas		$f(t)$	$F(s)$
		1	$\frac{1}{s}$
$\mathcal{L}[e^{at}f(t)](s) = F(s-a)$		$t^n, n = 1, 2, ...$	$\frac{n!}{s^{n+1}}$
$\mathcal{L}[f'](s) = s\mathcal{L}[f] - f(0)$		e^{at}	$\frac{1}{s-a}$
$\mathcal{L}[f''](s) = s^2\mathcal{L}[f] - sf(0) - f'(0)$		$t^n e^{at}, n = 1, 2, ...$	$\frac{n!}{(s-a)^{n+1}}$
$\mathcal{L}\left[\int_0^t f(\tau)d\tau)\right](s) = \frac{1}{s}\mathcal{L}[f]$		$\cos bt$	$\frac{s}{s^2+b^2}$
$\mathcal{L}[f * g](s) = \mathcal{L}[f](s)\mathcal{L}[g](s)$		$\sin bt$	$\frac{b}{s^2+b^2}$
$\mathcal{L}[f(t-c)u(t-c)](s) = e^{-cs}F(s),\ c > 0$		$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$\mathcal{L}[tf(t)](s) = -F'(s)$		$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$\mathcal{L}\left[\frac{f(t)}{t}\right](s) = \int_s^{\infty} F(\sigma)d\sigma$		$u(t-c), c > 0$	$\frac{e^{-cs}}{s}$
		$\delta(t-c), c > 0$	e^{-cs}

Fourier series and Fourier transform

- $2L$-periodic functions, real and complex form

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}) \sim \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}$$

  where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{in\pi x}{L}}dx$$

- Functions defined on the whole real line (need not be periodic)

$$\hat{f}(w) = \mathcal{F}[f](w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx,$$

$$f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}dw.$$

- Parseval's identities

$$\frac{1}{2L}\int_{-L}^{L}|f(x)|^2dx = \sum_{n=-\infty}^{\infty}|c_n|^2, \qquad \int_{-\infty}^{\infty}|f(x)|^2dx = \int_{-\infty}^{\infty}|\hat{f}(w)|^2dw$$

General formulas		$f(x)$	$\hat{f}(w)$
$\widehat{f'(x)} = iw\hat{f}(w)$		$\delta(x-a)$	$\frac{1}{\sqrt{2\pi}}e^{-iaw}$
$\widehat{f''(x)} = -w^2\hat{f}(w)$		$\begin{cases}1, & -b \le x \le b\\ 0, & \lvert x\rvert > b\end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin bw}{w}$
$f\widehat{(x-a)} = e^{-iaw}\hat{f}(w)$		$e^{-ax}u(x)$	$\frac{1}{\sqrt{2\pi}(a+iw)}$
$\hat{f}(w-b) = e^{\widehat{ibx}f(x)}$		$\frac{1}{x^2+a^2}$	$\sqrt{\frac{\pi}{2}}\frac{e^{-a\lvert w\rvert}}{a}$
$\widehat{f*g} = \sqrt{2\pi}\hat{f}\hat{g}$		e^{-ax^2}	$\frac{1}{\sqrt{2a}}e^{-w^2/(4a)}$

Complex numbers and analytic functions

- $e^{x+iy} = e^x(\cos y + i \sin y)$
- $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$, $\cosh z = \frac{e^z+e^{-z}}{2}$, $\sinh z = \frac{e^z-e^{-z}}{2}$
- Taylor and Laurent series of an analytic function

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz,$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}, \qquad b_n = \frac{1}{2\pi i}\oint_C f(z)(z-z_0)^{n-1} dz$$

Some useful integrals

$\int x \sin ax\, dx = \frac{1}{a^2}\sin ax - \frac{x}{a}\cos ax + C$

$\int x \cos ax\, dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax + C$

$\int x^2 \sin ax\, dx = \frac{2}{a^2} x \sin ax + \frac{2-a^2x^2}{a^3}\cos ax + C$

$\int x^2 \cos ax\, dx = \frac{2}{a^2} x \cos ax - \frac{2-a^2x^2}{a^3}\sin ax + C$

$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx) + C$

$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b\sin bx) + C$

$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$, $a > 0$

Some trigonometric identities

$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$

Some important series

- $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$, $\sum_{n=0}^{\infty} x^n$ diverges for $|x| \geq 1$.
- $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ for $x \in \mathbb{R}$.

Linear second order differential equations

Let r_1 and r_2 solve $r^2 + ar + b = 0$. Then

$$y''(x) + ay'(x) + by = 0$$

has general solution given by:

$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$	if	$r_1 \neq r_2, \quad r_1, r_2 \in \mathbb{R}$,
$y(x) = C_1 e^{r_1 x} + C_2 x e^{r_1 x}$	if	$r_1 = r_2, \quad r_1, r_2 \in \mathbb{R}$,
$y(x) = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$	if	$r_1 = \alpha + i\beta, \quad r_2 = \alpha - i\beta, \quad \alpha, \beta \in \mathbb{R}$.