MAGNUS EXPANSION IN GAUGE THEORIES

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Let us consider the initial value problem (IVP)

$$\dot{Y}(s) = A(s)Y(s), \quad Y(0) = Y_0, \tag{1}$$

where $A(s)$ is a matrix (operator) valued continuous function. For the moment, let us simplify the problem by assuming scalar functions, i.e. we ignore commutativity issues. Then the solution of the IVP (1) in terms of the exponential map, i.e. $Y(s) = \exp(\int_0^s A(x)dx)Y_0$. Expanding the exponential and making use of integration by parts immediately yields:

$$\exp\Big(\int_0^s A(x)dx\Big) = 1 + \int_0^s A(x_1)dx_1 \ + \int_0^s A(x_1)\int_0^{x_1} A(x_2)dx_2dx_1 \ + \cdots . \tag{2}$$

The righthand side is known as *Dyson–Chen series* (or time-ordered exponential) and corresponds to the integral equation associated to (1):

$$Y(s) = Y_0 + \int_0^s A(u)Y(u)du.$$

The Dyson–Chen series, i.e., righthand side of (2), holds when $A(t)$ is a $n \times n$ matrix valued continuous function. However, the exponential solution (on the lefthand side) changes drastically due to the non-commutative character of the problem.

Wilhelm Magnus proposed in 1954 [8] a differential equation for the matrix valued function $\Omega(s;A)$

$$\dot{\Omega}(s;A) = A(s) + \sum_{n>0} \frac{B_n}{n!} ad^{(n)}_{\int_0^s \dot{\Omega}(x;A)dx}(A(s)) = \frac{ad_{\Omega(s;A)}}{\exp(ad_{\Omega(s;A)}) - 1}(A(s)),$$

such that the solution of (1) is given by $X(s) = \exp(\int_0^s \dot{\Omega}(x;A)dx)Y_0$, $\Omega(0;A) = 0$. Here, the B_n are the Bernoulli numbers:

$$B_0 = 1, \ B_1 = -\frac{1}{2}, \ B_2 = \frac{1}{6}, \ B_4 = -\frac{1}{30}, \ldots \text{ and } B_{2k+1} = 0 \text{ for } k \geq 1,$$

and the usual notation for the n-fold iterated Lie bracket is used, i.e., $ad^{(n)}_U(W) := ad^{(n-1)}_U([U,W])$, $ad^{(0)}_U(W) = W$. Since its appearance, Magnus' seminal paper triggered much progress in both mathematics and physics. See e.g. [2, 3, 6, 9, 13]. Let us write down the first few terms of what is called *Magnus expansion*, $\Omega(s;\lambda A) = \sum_{n>0} \Omega_n(s;A)\lambda^n$:

$$\dot{\Omega}(s;\lambda A) = \lambda A(s) - \lambda^2 \frac{1}{2}\Big[\int_0^s A(x)dx, A(s)\Big] \tag{3}$$

$$+ \lambda^3 \frac{1}{4}\Big[\int_0^s \big[\int_0^y A(x)dx, A(y)\big]dy, A(s)\Big] + \lambda^3 \frac{1}{12}\Big[\int_0^s A(x)dx, \big[\int_0^s A(y)dy, A(s)\big]\Big] + \cdots , \tag{4}$$

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where we introduced a dummy parameter λ. Understanding this Lie series in full depth is a challenge.

Let $\pi : P \to Q$ be a (left) principal fiber bundle with structure group G and consider a G-connection on P whose connection one-form is given by $\mathcal{A} : TP \to \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. The holonomy of $\mathcal{A}$ is then defined as follows. For any closed loop $t \mapsto \sigma(t) \in Q$ in the base space Q, with $\sigma(0) = \sigma(1)$, we consider the horizontal lift $t \mapsto \tau(t) \in P$ to the total space P. It can be shown that $\tau(t)$ is the unique curve satisfying, for all t,

$$\pi(\tau(t)) = \sigma(t) \quad \text{and} \quad \mathcal{A}(\dot{\tau}(t)) = 0, \tag{5}$$

with initial condition $\tau(0) = p$, where p is a fixed point in P. In other words, τ projects down onto σ under π, and $\dot{\tau}$ is in the kernel of $\mathcal{A}$ (i.e. is a horizontal vector). We then define the *holonomy* of the curve τ as the unique group element $g_\tau \in G$ such that $\tau(1) = g_\tau \cdot \tau(0)$. See [7, 11] for some background.

Locally, we may assume that the fiber bundle is trivial, so that P is diffeomorphic to the product $Q \times G$, and the connection is given by $\mathcal{A} = g^{-1}dg + A_i(x)dx^i$, where the (x^i) are locally defined coordinates on Q. In this local trivialization, the curve τ is given by $\tau(t) = (\sigma(t), g(t))$, where $t \mapsto g(t)$ is a curve in G satisfying

$$\frac{dg}{dt} = -gA_i(x(t))\dot{x}^i(t), \tag{6}$$

where the $x^i(t)$ are the components of $\sigma(t)$ in the coordinate frame on Q. This equation can be solved using the Magnus expansion, and the holonomy associated to the curve τ is then given by $g_\tau = g(1)$. Using the condition that the loop σ is closed, $\sigma(0) = \sigma(1)$, as well as integrating by parts inside the expansion (3), the Magnus expansion can be reformulated, and in a local trivialization the holonomy is then given by

$$g_\tau = \exp \Omega_\tau, \quad \text{where} \quad \Omega_\tau = -\frac{1}{2}F_{ij}\int_\sigma dx^i dx^j + \frac{1}{3}\nabla_i F_{jk}\int_\sigma dx^i dx^j dx^k + \cdots, \tag{7}$$

where the F_{ij} are the local component functions of the *curvature* of $\mathcal{A}$, $\nabla_i F_{jk}$ is its covariant derivative, and the integrals are the moments of the curve $t \mapsto \sigma(t)$. This formula was derived by Burdick and Radford in [4]. However, it turns out that its precise description remains somewhat unclear. The idea is to use basic as well as more recent insights in the combinatorial and algebraic structure underlying the classical Magnus expansion [1, 5, 9, 12, 13] to obtain a more transparent description of (7).

THE AIM of this project is to study the Magnus expansion in the context of gauge theory [10]. This relies on the fact that the holonomy of a connection on a principal bundle can be written by means of a Magnus-like expression in a local chart. The additional structure present in the case of gauge theory (connections, curvatures, and covariant derivatives) then allows to rewrite this formula in a compact form (see (7)) involving only the curvature and its covariant derivatives.

Prerequisites: *interest in learning about the fascinating interplay between modern algebra, combinatorics and differential geometry, with a view toward applications in control theory. The key references are* [2, 4, 10].

References

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