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Solutions to exercise set 1

1 **a)** We can assume without loss of generality that $j = 1$. We can then write x as

$$x = (x_1, ..., x_n) = x_1(1, x_2/x_1, ..., x_n/x_1).$$

We calculate the p-norm

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} = |x_1| \left(1 + \left|\frac{x_2}{x_1}\right|^p + ... + \left|\frac{x_n}{x_1}\right|^p\right)^{1/p}.$$

We note that by assumption $|x_i/x_1 < 1$ for $i = 2, ..., n$. Therefore

$$1 + \left|\frac{x_2}{x_1}\right|^p + ... + \left|\frac{x_n}{x_1}\right|^p \to 1 \quad \text{as } p \to \infty.$$

We also note that for $a \geq 1$ we have $a^{1/p} \leq a$. Hence

$$\left(1 + \left|\frac{x_2}{x_1}\right|^p + ... + \left|\frac{x_n}{x_1}\right|^p\right)^{1/p} \to 1 \text{ as } p \to \infty.$$

Since $|x_1| = \max_{1 \leq i \leq n} |x_i|$ we have proven the statement.

b) Assume that there are $m \leq n$ indices (assumed to be $1, ..., m$) such that

$$|x_1| = ... = |x_m| = \max_{1 \leq i \leq n} |x_i|.$$

The p-norm is now

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} = |x_1| \left(1 + \left|\frac{x_2}{x_1}\right|^p + ... + \left|\frac{x_n}{x_1}\right|^p\right)^{1/p}.$$

Since $|x_1| = ... = |x_m|$ we have

$$\|x\|_p = |x_1| \left(m + \left|\frac{x_{m+1}}{x_1}\right|^p + ... + \left|\frac{x_n}{x_1}\right|^p\right)^{1/p}.$$

. We have

$$m + \left|\frac{x_{m+1}}{x_1}\right|^p + ... + \left|\frac{x_n}{x_1}\right|^p \to m \text{ as } p \to \infty,$$

and hence,

$$\left(m + \left|\frac{x_{m+1}}{x_1}\right|^p + ... + \left|\frac{x_n}{x_1}\right|^p\right)^{1/p} \to 1 \text{ as } p \to \infty.$$

Note that for any fixed $a > 0$ we have $a^{1/p} \to 1$ as $p \to \infty$.

c) Here is the code for generating the plots, which can be found at the end of the document.

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.linalg import norm

N = 101
X = np.linspace(-1.5,1.5, N)

$Plot D for p=1
plt.figure(figsize = (10,10))
for i in range(N):
for j in range(N):
if norm(np.array([X[i], X[j]]), 1) <= 1:
plt.plot(X[i],X[j], 'ko')

#Plot D for p=2
plt.figure(figsize = (10,10))
for i in range(N):
for j in range(N):
if norm(np.array([X[i], X[j]]), 2) <= 1:
plt.plot(X[i],X[j], 'ko')

#Plot D for p=4
plt.figure(figsize = (10,10))
for i in range(N):
for j in range(N):
if norm(np.array([X[i], X[j]]), 4) <= 1:
plt.plot(X[i],X[j], 'ko')

#Plot D for p=inf
plt.figure(figsize = (10,10))
for i in range(N):
for j in range(N):
if norm(np.array([X[i], X[j]]), np.inf) <= 1:
plt.plot(X[i],X[j], 'ko')
```

2 Since $p_1 = 1$ already satisfies $\|p_1\| = 1$ we do not need to normalize it. Hence, $q_1 = p_1 = 1$. In order to calculate q_2 we first need to calculate

$$\langle p_2, q_1\rangle = \int_0^1 x \, \mathrm{d}x = \tfrac{1}{2}.$$

We then calculate

$$r_2 = p_2 - \langle p_2, q_1\rangle q_1 = x - \tfrac{1}{2} \cdot 1 = x - \tfrac{1}{2}.$$

We calculate the norm

$$\|r_2\| = \left(\int_0^1 r_2(x)^2 \, \mathrm{d}x\right)^{1/2} = \tfrac{1}{2\sqrt{3}}.$$

From this we get $q_2 = r_2/\|r_2\| = 2\sqrt{3}(x - \frac{1}{2})$. We likewise need to calculate

$$\langle p_3, q_1 \rangle = \int_0^1 x^2 \,\mathrm{d}x = \tfrac{1}{3},$$

$$\langle p_3, q_2 \rangle = \int_0^1 2\sqrt{3}x^2(x - \tfrac{1}{1}) \,\mathrm{d}x = \tfrac{1}{2\sqrt{3}}.$$

This allows us to calculate

$$r_3 = p_3 - \langle p_3, q_1 \rangle q_1 - \langle p_3, q_2 \rangle q_2 = x^2 - x + \tfrac{1}{6}.$$

We calculate the norm

$$\|r_3\| = \big((x^2 - x + \tfrac{1}{6})^2 \,\mathrm{d}x\big)^{1/2} = \tfrac{1}{\sqrt{180}} = \tfrac{1}{6\sqrt{5}}.$$

This means that finally,

$$q_1 = 1, \quad q_2 = 2\sqrt{3}(x - \tfrac{1}{2}), \quad q_3 = 6\sqrt{5}(x^2 - x + \tfrac{1}{6}).$$

3 The main tool to solve this exercise is to use the trigonometric formulas in the hint, and the fact that

$$\int_0^{2\pi} \sin(nx) \,\mathrm{d}x = 0$$

and

$$\int_0^{2\pi} \cos(nx) \,\mathrm{d}x = \begin{cases} 2\pi, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

The answers are therefore

a) 2π

b) π

c) 0

d) 0.

4 **a)** We se that $\|f\|_C = \max_{x\in[0,1]} |f(x)| = 0$ if and only if $f(x) = 0$ for all $x \in [0, 1]$, that is when $f = 0$. We also have for any $a \in \mathbb{R}$

$$\|af\|_C = \max_{x\in[0,1]} |af(x)| = \max_{x\in[0,1]} |a| \cdot |f(x)| = |a| \cdot \max_{x\in[0,1]} |f(x)| = |a| \cdot \|f\|_C.$$

Finally, let f and g have the global maximas x_f and x_g. Then

$$\max_{x\in[0,1]} |f(x) + g(x)| \leq \max_{x\in[0,1]} |f(x)| + |g(x)| \leq |f(x_f)| + |g(x_g)| = \|f\|_C + \|g\|_C.$$

Therefore, $\|\cdot\|_C$ satisfies all requirements of a norm.

b) For $\|\cdot\|_\star$ all calculations are the same (with $[0, 0.5]$ replacing $[0, 1]$), except the first property. This can be seen by the function

$$\hat{f}(x) = \begin{cases} 0, & x \leq 0.5, \\ x - 0.5, & x > 0.5. \end{cases}$$

This function is clearly not 0 but satisfies $\|\hat{f}\|_\star = 0$. Hence, $\|\cdot\|_\star$ is not a norm. It is however a semi-norm, which is defined to satisfy the assumptions of a norm except that $\|f\| = 0 \Rightarrow f = 0$.

Plots of unit-discs

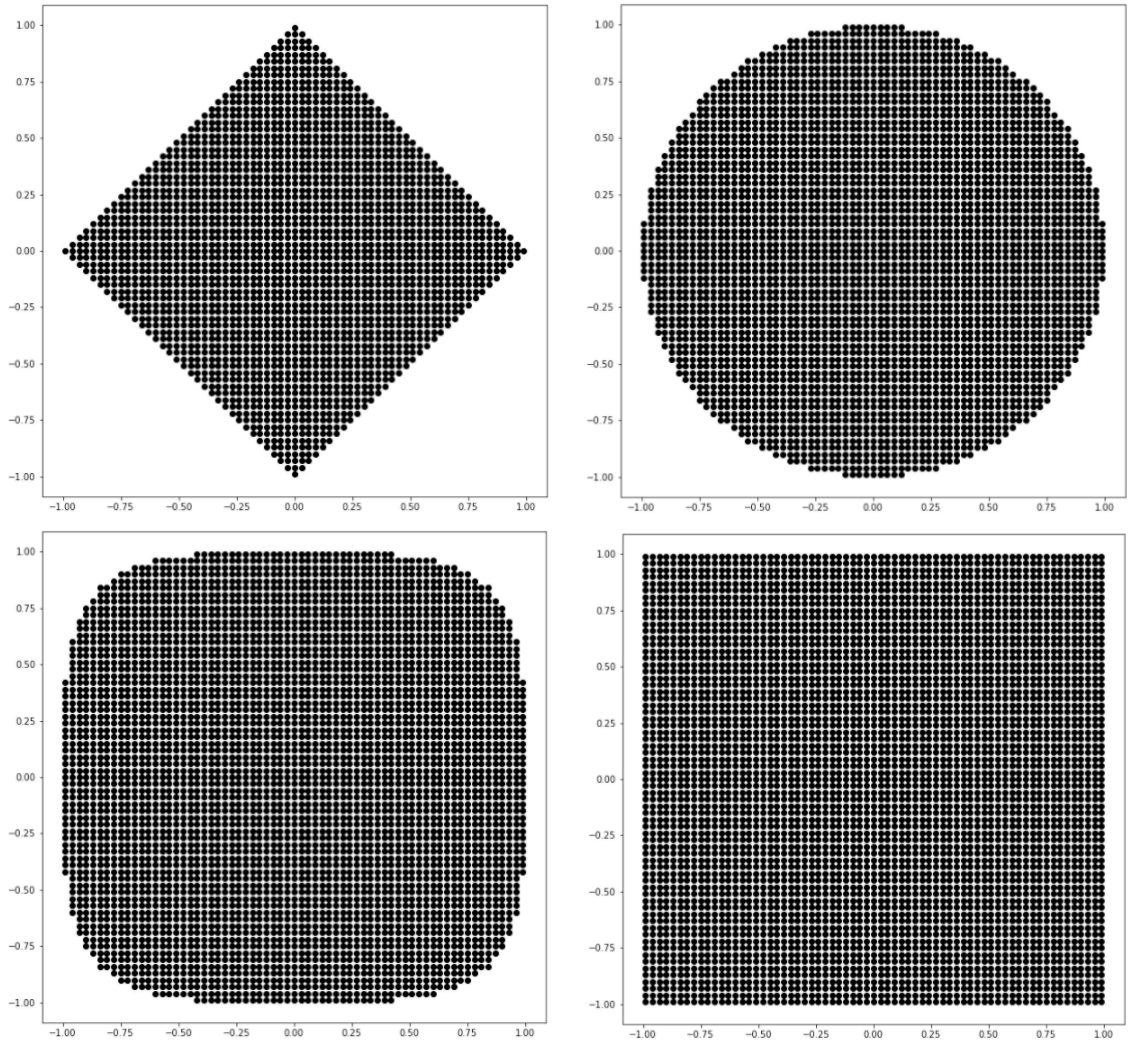

Figure 1: Plots of unit-discs for $p = 1$ (top left), $p = 2$ (top right), $p = 4$ (bottom left) and $p = \infty$ (bottom right).