Exercise 5

Problems from the textbook:

7.3 Given a random sample $X_1, \dots, X_n$ from a population with pdf $f(x|\theta)$, show that maximizing the likelihood function, $L(\theta|\mathbf{x})$, as a function of θ is equivalent to maximizing $\log L(\theta|\mathbf{x})$.

7.6 Let $X_1, \dots, X_n$ be a random sample from the pdf

$$f(x|\theta) = \theta x^{-2}, \quad 0 < \theta \le x < \infty.$$

(a) What is a sufficient statistic for θ?
(b) Find the MLE of θ.
(c) Find the method of moments estimator of θ.

7.9 Let $X_1, \dots, X_n$ be iid with pdf

$$f(x|\theta) = \frac{1}{\theta}, \quad 0 \le x \le \theta, \quad \theta > 0.$$

Estimate θ using both the method of moments and maximum likelihood. Calculate the means and variances of the two estimators. Which one should be preferred and why?

Info for 7.22:

Example 7.2.16 (Normal Bayes estimators) Let $X \sim \mathrm{n}(\theta, \sigma^2)$, and suppose that the prior distribution on θ is $\mathrm{n}(\mu, \tau^2)$. (Here we assume that σ^2, μ, and τ^2 are all known.) The posterior distribution of θ is also normal, with mean and variance given by

$$\mathrm{E}(\theta|x) = \frac{\tau^2}{\tau^2+\sigma^2}x + \frac{\sigma^2}{\sigma^2+\tau^2}\mu,$$

(7.2.10)

$$\mathrm{Var}\,(\theta|x) = \frac{\sigma^2\tau^2}{\sigma^2+\tau^2}.$$

7.22 This exercise will prove the assertions in Example 7.2.16, and more. Let $X_1, \dots, X_n$ be a random sample from a $\mathrm{n}(\theta, \sigma^2)$ population, and suppose that the prior distribution on θ is $\mathrm{n}(\mu, \tau^2)$. Here we assume that σ^2, μ, and τ^2 are all known.

(a) Find the joint pdf of $\bar{X}$ and θ.
(b) Show that $m(\bar{x}|\sigma^2, \mu, \tau^2)$, the marginal distribution of $\bar{X}$, is $\mathrm{n}(\mu, (\sigma^2/n) + \tau^2)$.
(c) Show that $\pi(\theta|\bar{x}, \sigma^2, \mu, \tau^2)$, the posterior distribution of θ, is normal with mean and variance given by (7.2.10).

7.24 Let $X_1, \dots, X_n$ be iid Poisson(λ), and let λ have a gamma(α, β) distribution, the conjugate family for the Poisson.

(a) Find the posterior distribution of λ.
(b) Calculate the posterior mean and variance.

Discuss: What is the difference between the likelihood function and the likelihood statistic? Demonstrate the difference by an example. Remember: A statistic is a function of the data!

From previous exams:

H2020, problem 1b, c

H2019, problem 3b

V2009, problem 2c (a and b are also good practice)