TMA4285 Time series models
Solution to exercise 7, autumn 2020

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Problem 6.1

The difference equations are satisfied if $(1-B)^d(A_0 + A_1 t + \cdots + A_{d-1}t^{d-1}) = 0$.

$(1-B)t^q$ is a polynomial of degree $q-1$, and $(1-B)c = c - c = 0$. It follows that

$$(1-B)^d(A_0 + A_1 t + \cdots + A_{d-1}t^{d-1}) = (1-B)^d A_0 + (1-B)^d A_1 t + \dots (1-B)^d A_{d-1}t^{d-1},$$

and$(1-B)^d A_0 = 0, (1-B)^d t^q = 0$ for $q = 1, ..., d-1$ from which the result follows.

Problem 6.2

We want to verify the representation given in (6.3.4). We start with the equation given in (6.3.4) and insert $\phi_0^*, \phi_1^*, \phi_j^*$ and $\nabla X_t = X_t - X_{t-1}$,

$$X_t - X_{t-1} = \phi_0^* + \phi_1^* X_{t-1} + \phi_2^*(X_{t-1} - X_{t-2}) + \cdots + \phi_p^*(X_{t-p+1} - X_{t-p}) + Z_t$$

$$X_t = \mu(1 - \phi_1 - \cdots - \phi_p) + X_{t-1} + (\sum_{i=1}^{p} \phi_i - 1)X_{t-1} - \sum_{i=2}^{p} \phi_i(X_{t-1} - X_{t-2})$$

$$- \sum_{i=3}^{p} \phi_i(X_{t-2} - X_{t-3}) - \cdots - \sum_{i=p-1}^{p} \phi_i(X_{t-p} - X_{t-p-1}) - \sum_{i=p}^{p} \phi_i(X_{t-p+1} - X_{t-p}) + Z_t$$

Terms will cancel out such that we are left with

$$X_t - \mu = -\mu(\phi_1 + \cdots + \phi_p) + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + Z_t$$
$$X_t - \mu = \phi_1(X_{t-1} - \mu) + \cdots + \phi_p(X_{t-p} - \mu) + Z_t$$

which is the what we wanted to verify.

Problem 6.11

a)

The first steps in identifying SARIMA models for a (possibly transformed) data set are to find d and D so as to make the differenced observations stationary in appearance. The differencing at lag 12 and lag 1, suggests $d = D = 1$ and $s = 12$. Since the ACF at lags of 12 decays slowly, this suggests a seasonal AR part, probably $P = 1$ and $Q = 0$. Using example 1.4.5, we get that $\Phi = 0.8$. The ACF next to lags of 12 has cutoff after 1 lag. This suggests a MA part for the non-seasonal part, $q = 1$ and $p = 0$. From example 1.4.4 we see that θ is given by

$$0.4 = \frac{\theta}{1 + \theta^2}$$

Solving this gives $\theta_1 = 2$ and $\theta_2 = 0.5$. Choosing $\theta = 0.5$ gives an invertible ARMA process for the differenced series.

b)

We want to express the one- and twelve-step ahead linear predictors $P_n X_{n+1}$ and $P_n X_{n+12}$ for large n.

The linear predictors are given by eq (6.5.11) in Brockwell and Davis

$$P_n X_{n+h} = P_n Y_{n+h} + \sum_{j=1}^{d+Ds} a_j P_n X_{n+h-j},$$

where $P_n Y_{n+h}$ is the best linear predictor of the ARMA process $\{Y_t\}$ and $P_n X_{n+h}$ can be computed recursively.

We start with $P_n X_{n+1}$. The ARMA process $\{Y_t\}$ is defined by

$$\phi(B)\Phi(B^s)Y_t = \theta(B)\Theta(B^s)Z_t$$

and with our values from a), we get

$$(1 - \Phi B^{12})Y_t = (1 - \theta B)Z_t \tag{1}$$

which is an ARMA(12,1) with $\Phi_1 = ... = \Phi_{11} = 0$, $\Phi_{12} = \Phi$ and θ from a).

From section 3.3, we find

$$P_n Y_{n+1} = \Phi Y_{n-11} + \theta_{n,1}(Y_n - \hat{Y}_n)$$

$\theta_{n,1}$ can be found from the innovations algorithm with κ as in (3.3.3).

Then we get

$$P_n X_{n+1} = \Phi Y_{n-11} + \theta_{n,1}(Y_n - \hat{Y}_n) + \sum_{j=1}^{13} a_j X_{n+h-j} \tag{2}$$

Next, we find $P_n X_{n+h}$. For the ARMA process $\{Y_t\}$ now need

$$P_n Y_{n+12} = \Phi P_n Y_{n+11} + \theta_{n+11,12}(Y_n - \hat{Y}_n)$$

Again $\theta_{n+11,12}$ can be found from the innovations algorithm with κ as in (3.3.3). We get

$$P_n X_{n+12} = \Phi P_n Y_{n+11} + \theta_{n+11,12}(Y_n - \hat{Y}_n) + \sum_{j=1}^{13} a_j P_n X_{n+12-j} \tag{3}$$

$P_n X_{n+12-j}$ can be computed recursively.

The a_j in equation (2) and (3) can be found by comparing (6.5.10) in the book using $h = 0$

$$X_t = Y_t + \sum_{j=0}^{13} a_j X_{t-j}$$

with our equation for X_t. The equation for X_t is found solving $Y_t = (1-B)(1-B^{12})$ for X_t. Doing this gives

$$X_t = Y_t + X_{t-1} + X_{t-12} - X_{t-13}$$

From this we see that $a_1 = a_{12} = 1$, $a_{13} = -1$ and the rest must be zero.

c) The mean square errors of the predictors are given by

$$\sigma_n^2(h) = \sum_{j=0}^{h-1} \psi_j \sigma^2$$

where $\psi_1, ..., \psi_j$ can be computed from

$$\phi(z) = \frac{\theta(z)\Theta(z^s)}{\phi(z)\Phi(z^s)(1-z)^d(1-z^s)^D}$$

In our case this equation becomes

$$\phi(z) = \frac{1 - \theta z}{(1 - \Phi z^{12})(1-z)(1-z^{12})}$$

Solving this gives $\psi_0 = 1$ and $\psi_1 = \cdots = \psi_{11} = 1 - \theta$. We finally get

$$\sigma_n^2(1) = \psi_0^2 \sigma^2 = \sigma^2$$
$$\sigma_n^2(12) = \sum_{j=0}^{11} \psi_j^2 \sigma^2 = \sigma^2 + 11\sigma^2(1-\theta)^2$$

Chapter 6

Problem 6.5. The best linear predictor of Y_{n+1} in terms of $1, X_0, Y_1, \ldots, Y_n$ i.e.

$$\hat{Y}_{n+1} = a_0 + cX_0 + a_1Y_1 + \cdots + a_nY_n,$$

must satisfy the orthogonality relations

$$\begin{aligned} \mathrm{Cov}(Y_{n+1} - \hat{Y}_{n+1}, 1) &= 0 \\ \mathrm{Cov}(Y_{n+1} - \hat{Y}_{n+1}, X_0) &= 0 \\ \mathrm{Cov}(Y_{n+1} - \hat{Y}_{n+1}, Y_j) &= 0, \quad j = 1, \ldots, n. \end{aligned}$$

The second equation can be written as

$$\mathrm{Cov}(Y_{n+1} - \hat{Y}_{n+1}, X_0) = \mathbb{E}[(Y_{n+1} - a_0 + cX_0 + a_1Y_1 + \cdots + a_nY_n)X_0] = c\mathbb{E}[X_0^2] = 0$$

so we must have $c = 0$. This does not effect the other equations since $\mathbb{E}[Y_jX_0] = 0$ for each j.

Problem 6.6. Put $Y_t = \nabla X_t$. Then $\{Y_t : t \in \mathbb{Z}\}$ is an AR(2) process. We can rewrite this as $X_{t+1} = Y_t + X_{t-1}$. Putting $t = n + h$ and using the linearity of the projection operator P_n gives $P_nX_{n+h} = P_nY_{n+h} + P_nX_{n+h-1}$. Since $\{Y_t : t \in \mathbb{Z}\}$ is AR(2) process we have $P_nY_{n+1} = \phi_1Y_n + \phi_2Y_{n-1}$, $P_nY_{n+2} = \phi_1P_nY_{n+1} + \phi_2Y_n$ and iterating we find $P_nY_{n+h} = \phi_1P_nY_{n+h-1} + \phi_2P_nY_{n+h-2}$. Let $\phi^*(z) = (1-z)\phi(z) = 1 - \phi_1^*z - \phi_2^*z^2 - \phi_3^*z^3$. Then

$$(1-z)\phi(z) = 1 - \phi_1z - \phi_2z - z + \phi_1z^2 + \phi_2z^3,$$

i.e. $\phi_1^* = \phi_1 + 1$, $\phi_2^* = \phi_2 - \phi_1$ and $\phi_3^* = -\phi_2$. Then

$$P_nX_{n+h} = \sum_{j=1}^{3} \phi_j^* X_{n+h-j}.$$

This can be verified by first noting that

$$\begin{aligned} P_nY_{n+h} &= \phi_1P_nY_{n+h-1} + \phi_2P_nY_{n+h-2} \\ &= \phi_1(P_nX_{n+h-1} - P_nX_{n+h-2}) + \phi_2(P_nX_{n+h-2} - P_nX_{n+h-3}) \\ &= \phi_1P_nX_{n+h-1} + (\phi_2 - \phi_1)P_nX_{n+h-2} - \phi_2P_nX_{n+h-3}. \end{aligned}$$

and then

$$\begin{aligned} P_nX_{n+h} &= P_nY_{n+h} + P_nX_{n+h-1} \\ &= (\phi_1 + 1)P_nX_{n+h-1} + (\phi_2 - \phi_1)P_nX_{n+h-2} - \phi_2P_nX_{n+h-3} \\ &= \phi_1^*P_nX_{n+h-1} + \phi_2^*P_nX_{n+h-2} + \phi_3^*P_nX_{n+h-3}. \end{aligned}$$

Hence, we have

$$g(h) = \begin{cases} \phi_1^*g(h-1) + \phi_2^*g(h-2) + \phi_3^*g(h-3), & h \geq 1, \\ X_{n+h}, & h \leq 0. \end{cases}$$

We may suggest a solution of the form $g(h) = a + b\xi_1^{-h} + c\xi_2^{-h}$, $h > -3$ where ξ_1 and ξ_2 are the solutions to $\phi(z) = 0$ and $g(-2) = X_{n-2}$, $g(-1) = X_{n-1}$ and $g(0) = X_n$. Let us first find the roots ξ_1 and ξ_2.

$$\phi(z) = 1 - 0.8z + 0.25z^2 = 1 - \frac{4}{5}z + \frac{1}{4}z^2 = 0 \Rightarrow z^2 - \frac{16}{5}z + 4 = 0.$$

We get that $z = 8/5 \pm \sqrt{(8/5)^2 - 4} = (8 \pm 6i)/5$. Then $\xi_1^{-1} = 5/(8+6i) = \cdots = 0.4 - 0.3i$ and $\xi_2^{-1} = 0.4 + 0.3i$. Next we find the constants a, b and c by solving

$$\begin{aligned} X_{n-2} &= g(-2) = a + b\xi_1^{-2} + c\xi_2^{-2}, \\ X_{n-1} &= g(-1) = a + b\xi_1^{-1} + c\xi_2^{-1}, \\ X_n &= g(0) = a + b + c. \end{aligned}$$

Note that $(0.4 - 0.3i)^2 = 0.07 - 0.24i$ and $(0.4 + 0.3i)^2 = 0.07 + 0.24i$ so we get the equations

$$\begin{aligned} X_{n-2} &= a + b(0.07 - 0.24i) + c(0.07 + 0.24i), \\ X_{n-1} &= a + b(0.4 - 0.3i) + c(0.4 + 0.3i), \\ X_n &= a + b + c. \end{aligned}$$

Let $a = a_1 + a_2 i$, $b = b_1 + b_2 i$ and $c = c_1 + c_2 i$. Then we split the equations into a real part and an imaginary part and get

$$\begin{aligned} X_{n-2} &= a_1 + 0.07b_1 + 0.24b_2 + 0.07c_1 - 0.24c_2, \\ X_{n-1} &= a_1 + 0.4b_1 + 0.3b_2 + 0.4c_1 - 0.4c_2, \\ X_n &= a_1 + b_1 + c_1, \\ 0 &= a_2 + 0.07b_2 - 0.24b_1 + 0.07c_2 + 0.24c_1, \\ 0 &= a_2 + 0.4b_2 - 0.3b_1 + 4c_2 + 0.3c_1, \\ 0 &= a_2 + b_2 + c_2. \end{aligned}$$

We can write this as a matrix equation by

$$\begin{pmatrix} 1 & 0 & 0.07 & 0.24 & 0.07 & -0.24 \\ 1 & 0 & 0.4 & 0.3 & 0.4 & -0.3 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -0.24 & 0.07 & 0.24 & 0.07 \\ 0 & 1 & -0.3 & 0.4 & 0.3 & 0.4 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} X_{n-2} \\ X_{n-1} \\ X_n \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which has the solution $a = 2.22X_n - 1.77X_{n-1} + 0.55X_{n-2}$, $b = \bar{c} = -1.1X_{n-2} + 0.88X_{n-1} + 0.22X_n + (-2.22X_{n-2} + 3.44X_{n-1} - 1.22X_n)i$.

GT Exercises

Exercise 6

a The general expression for a SARIMA$(p,d,q)\times(P,D,Q)_s$ model is:

$$\phi(B)\Phi(B^s)Y_t = \theta(B)\Theta(B^s)Z_t, \qquad \{Z_t\}\sim WN(0,\sigma^2) \qquad (1)$$

with Y_t the differenced time series $Y_t = (1-B)^d(1-B^s)^D X_t$ a causal ARMA process. Hence, (1) becomes:

$$\begin{aligned} \phi(B)\Phi(B^s)(1-B)^d(1-B^s)^D X_t &= \theta(B)\Theta(B^s)Z_t, \qquad \{Z_t\}\sim WN(0,\sigma^2) \\ \phi_S(B)\phi_N(B)X_t &= \theta(B)\Theta(B^s)Z_t \end{aligned} \qquad (2)$$

Note that $\phi_N(z) = (1-z)^d(1-z^s)^D$ has zeros only in $S=\{z:|z|=1\}$. On the other hand, given that the process is causal, $\phi_S(z) = (1-\phi_1(z)-\phi_2 z^2-\ldots-\phi_p z^p)(1-\Phi_1 z^s-\Phi_2 z^{2s}-\ldots-\Phi_P z^{Ps})$ has no zeros in $S=\{z:|z|=1\}$ since all of its zeros satisfy $|z|>1$.

b Starting from (2), if we assume $\phi_N(B)X_t$, then we get:

$$\phi_S(B)Y_t = \theta(B)\Theta(B^s)Z_t$$

$$(1-\phi_1 B-\cdots-\phi_p B^p)(1-\Phi_1 B^s-\cdots-\Phi_P B^{Ps})Y_t = \theta(B)\Theta(B^s)Z_t$$

$$(1-\phi_1 B-\cdots-\phi_p\Phi_{Ps}B^{p+Ps})Y_t = \theta(B)\Theta(B^s)Z_t$$

$$(1-\phi_1 B-\cdots-\phi_p\Phi_{Ps}B^{p+Ps})Y_t = (1+\theta_1 B+\cdots+\theta_q\Theta_{Qs}B^{q+Qs})Z_t$$

Thus, Y is an $ARMA(p+Ps, q+Qs)$ process with some coefficients constrained to be zero. In the general case with $E(X_t)=\mu^*$,

$$\begin{aligned} E(Y_t) &= (1-\phi_1 B-\cdots-\phi_p\Phi_{Ps}B^{p+Ps})E(X_t) \\ &= \mu^*(1-\phi_1-\cdots-\phi_p\Phi_{Ps}) \\ &= \mu \end{aligned}$$

c From part b we know

$$\phi_N(B)X = Y \tag{3}$$

with Y an ARMA(p+Ps,q+Qs) process. Based on (3) we can express Y_t as

$$\begin{aligned} Y_t &= (1-B)^d(1-B^s)^D X_t \\ &= X_t + \sum_{j=1}^{N} a_j X_{t-j}, \quad t = 1,\ldots,n \end{aligned}$$

That is, any linear combination of $\{X_{-N+1},\ldots,X_0,Y_1,\ldots,Y_n\}$ can be expressed as a linear combination of $\{X_{-N+1},\ldots,X_0,X_1,\ldots,X_n\}$. Similarly,

$$X_t = Y_t - \sum_{j=1}^{N} a_j X_{t-j}, \quad t = 1,\ldots,n$$

Hence, any linear combination of $\{X_{-N+1},\ldots,X_0,X_1,\ldots,X_n\}$ can be expressed as a linear combination of $\{X_{-N+1},\ldots,X_0,Y_1,\ldots,Y_n\}$. Thus, the best linear predictor of X_{n+1} based on $\{X_{-N+1},\ldots,X_0,X_1,\ldots,X_n\}$ given by the projection of X_{n+1} on $\bar{sp}\{X_{-N+1},\ldots,X_0,X_1,\ldots,X_n\}$ is the same as the best linear predictor of X_{n+1} based on $\{X_{-N+1},\ldots,X_0,Y_1,\ldots,Y_n\}$ since

$$\bar{sp}\{X_{-N+1},\ldots,X_0,X_1,\ldots,X_n\} = \bar{sp}\{X_{-N+1},\ldots,X_0,Y_1,\ldots,Y_n\}$$

d If d,D and s are known, then from $\{X_{-N+1},\ldots,X_n\}$ we can compute

$$Y_t = \phi_N(B)X_t \qquad t = 1,\ldots,n$$

Now, based only on Y_t, we are able to fit the $ARMA(p+Ps,q+Qs)$ process

$$\phi_S(B)Y = \theta(B)Z$$

through the innovations algorithm outlined in section 5.1.3 of the book, which depends on

$$\theta_{n,n-k} = \nu_k^{-1}\left(\kappa(n+1,k+1) - \sum_{j=0}^{k-1}\theta_{k,k-j}\theta_{n,n-j}\nu_j\right)$$

which depends only on the ACVF of Y, $\gamma_Y(k)$, known since the orders p, q, P and Q are known.

e The set of observations $\{X_{-N+1}, \ldots, X_0\}$ is necessary for computing the set of differenced observations Y. Let's remind that in general

$$Y_t = X_t + \sum_{j=1}^{N} a_j X_{t-j}, \quad t = 1, \ldots, n$$

In addition to these observations are necessary for prediction since the best linear predictor $P_n X_{n+h}$ is found as the projection of X_{n+h} on the closed span of $\{X_{-N+1}, \ldots, X_0, X_1, \ldots, X_n\}$ or equivalently the closed span of $\{X_{-N+1}, \ldots, X_0, Y_1, \ldots, Y_n\}$ as shown in part c.

f If we let Z be Gaussian, then the ARMA model associated to Y has likelihood

$$(2\pi\sigma^2)(r_0 \cdot r_1 \cdots r_{n-1})^{-1/2} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{j=1}^{n} (Y_j - \hat{Y}_j)^2 / r_{j-1} \right\}$$

Now, if we take into account that $r_0, \ldots, r_{n+1}$ and $\hat{Y}_j$ are given by:

$$r_{i-1} = \frac{1}{\sigma^2} E(Y_i - \hat{Y}_i)^2$$

$$\hat{Y}_{i+1} = \begin{cases} \sum_{j=1}^{i} \theta_{ij}(Y_{i+1-j} - \hat{Y}_{i+1-j}) & 1 \leq i < m = \max(p + Ps, q + Qs) \\ \phi_1 Y_i + \cdots + \phi_{p+Ps} Y_{i+1-p-Ps} + \sum_{j=1}^{q+Qs} \theta_{ij}(Y_{i+1-j} - \hat{Y}_{i+1-j}) & i \geq m \end{cases}$$

with $\theta_{n,n-k} = \nu_k^{-1}\left(\kappa(n+1, k+1) - \sum_{j=0}^{k-1} \theta_{k,k-j}\theta_{n,n-j}\nu_j\right)$, where $\nu_k = E(Y_{k+1} - \hat{Y}_{k+1})^2$. Hence, given that all the terms involved in the likelihood depend on $\{Y_1, \ldots, Y_n\}$, we conclude the information of the model is contained in $\{Y_1, \ldots, Y_n\}$.