

Problem 1:

a) Scheme:

$$\frac{1}{k}(U_i^{n+1} - U_i^n) = \frac{a}{h^2}(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}) - U_i^{n+1}$$

$$\Rightarrow \begin{cases} U_i^{n+1} = U_i^n + a \frac{k}{h^2} (U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}) - k \cdot U_i^{n+1}, & i = 1, \dots, M-1 \\ U_0^n = U_M^n = 0 \end{cases}$$

Local truncation error, $\tilde{\tau}_i^n$: Let $u_i^n = u(x_i, t_n)$

$$k \cdot \tilde{\tau}_i^n = U_i^{n+1} - U_i^n - a \frac{k}{h^2} (u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}) - k u_i^{n+1}$$

For simplicity, do the expansion around (x_i, t_{n+1}) :

$$\begin{aligned} k \cdot \tilde{\tau}_i^n &= k \cdot \partial_t u + \frac{1}{2} k^2 \partial_t^2 u + \mathcal{O}(k^3) \\ &\quad - a \cdot k \left[\partial_x^2 u + h^2 \cdot \frac{1}{12} \partial_x^4 u + \mathcal{O}(h^6) \right] + k \cdot u \end{aligned}$$

so, using that $\partial_t u - \partial_x^2 u + u = 0$ we get

$$\tilde{\tau}_i^n = \frac{1}{2} k \partial_t^2 u - h^2 \frac{a}{12} \partial_x^4 u + \mathcal{O}(k^2) + \mathcal{O}(h^4)$$

all evaluated in (x_i, t_{n+1}) , or in (x_i, t_n) .

The difference will only appear in the higher order terms.

b) The scheme is consistent ($\tilde{\tau}_i^n \rightarrow 0$), so according to the Lax equivalence theorem, $h, k \rightarrow 0$ it is sufficient to prove stability.

Let $r = k/h^2$. The scheme can be written as

$$U_i^{n+1} - r(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}) + k \cdot U_i^{n+1} = U_i^n$$

Using $U^n = [U_1^n, \dots, U_{M-1}^n]^T$, this becomes

$$A \cdot U^{n+1} = U^n \quad \Rightarrow \quad U^{n+1} = C \cdot U^n \quad \text{with } C = A^{-1} \text{ and}$$

where $A = (1+k)I + r \cdot T$, $T = \text{tridiag} \{-1, 2, -1\}$

We know! The scheme is stable if $\rho(C) \leq 1$

The eigenvalues are:

$$\lambda_j(T) = 4 \sin^2 \left(\frac{j\pi}{2M} \right) \quad (\text{from the note})$$

$$\lambda_j(A) = (1+k) + r \cdot \lambda_j(T) = 1+k + 4r \sin^2\left(\frac{j\pi}{2M}\right)$$

and

$$\lambda_j(C) = \frac{1}{\lambda_j(A)} = \frac{1}{1+k+4r \sin^2\left(\frac{j\pi}{2M}\right)}$$

and clearly

$$0 < \lambda_j(C) < 1 \Rightarrow \rho(C) < 1$$

and the scheme is stable, and thus convergent.

Problem 2:

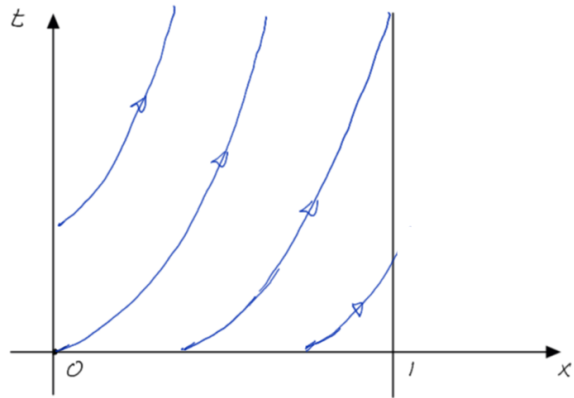
a) The characteristics are the solution of the equations

$$\frac{dx}{dt} = \frac{1}{(1+t)^2} \quad x(t^*) = x^*$$

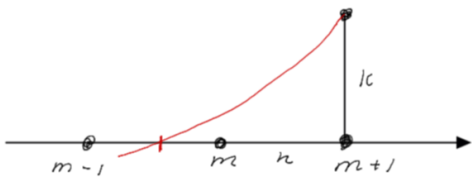
The solution is

$$x(t) = x^* + \frac{1}{1+t^*} - \frac{1}{1+t}$$

The characteristics are moving from left to right, so we should include boundary conditions at $x=0$.



b) We notice that the velocity $\frac{1}{(1+t)^2}$ becomes smaller as t increases, so for the CFL condition to be satisfied everywhere, it suffices to have it satisfied for the first step. The requirement then is that the characteristic through (x_{m+1}, k) should pass through the interval (x_{m-1}, x_{m+1}) at $t=0$ (see the figure).



Computational stencil

The condition becomes:

$$x(0) = x_{m+1} + \frac{1}{1+k} - 1 \in [x_{m-1}, x_{m+1}]$$

$$\Rightarrow x_{m+1} + \frac{1}{1+k} - 1 \geq x_{m-1}$$

$$\Rightarrow 2h + \frac{1}{1+k} \geq 1 \Rightarrow k \leq \frac{2h}{1-2h}$$

If $h = 1/4$, $k \leq 1$.

Problem 3:

This is not a complete solution to this problem, but some indications on what can be done:

If we do not know much about w and a , it may be a good idea to go for 2. order approximations:

$$w(u)u_x \rightarrow w(U_i) \cdot \frac{1}{2h} (U_{i+1} - U_{i-1})$$

$$(a(u)u_x)_x \rightarrow \frac{1}{h^2} (a_{i+1/2} U_{i+1} - (a_{i+1/2} + a_{i-1/2})U_i + a_{i-1/2} U_i)$$

where one of the following can be used:

$$a_{i+1/2} = \frac{1}{2} (a(U_{i+1}) + a(U_i))$$

or

$$a_{i+1/2} = a\left(\frac{1}{2}(U_{i+1} + U_i)\right)$$

Boundary conditions:

$$x = 0 : U_0 = 0$$

$$x = 1 : u_x + u = 0 \rightarrow \frac{U_{M+1} - U_{M-1}}{2h} + U_M = 0$$

$$\Rightarrow U_{M+1} = U_{M-1} - 2hU_M$$

which is inserted into the difference formula (not written out here) for $i = M$.

Altogether, this ends up in a system of ODEs.

$$\dot{U} = F(U), \text{ where } U = [U_1, \dots, U_M].$$

Since we have a diffusion term, we would choose an implicit (A-stable) scheme to solve the ODE.

The trapezoidal rule (Crank-Nicolson) is of order 2, so that may be a good choice:

$$U^{n+1} = U^n + \frac{h}{2} [F(U^{n+1}) + F(U^n)].$$

which altogether creates a system of nonlinear equations which has to be solved for each step.

Other considerations:

If the diffusion term is small, you may want to use an upwind scheme ($w > 0$):

$$w(u)u_x \rightarrow w(U_i) \frac{U_i - U_{i-1}}{h}. \quad (\text{order 1})$$

In this case, it may also be possible to use an explicit scheme, but stability may be an issue.

Problem 4: $a > 0$

Scheme: $U_m^{n+1} = U_m^n - p \cdot (U_m^n - U_{m-1}^n)$

Let $U_m^n = \xi^n e^{i\beta x_m}$, and we get

$$\xi = 1 - p(1 - e^{-i\beta h}) = 1 - p + p(\cos(\beta h) - i \sin(\beta h))$$

and

$$|\xi|^2 = (1 - p + p \cos(\beta h))^2 + p^2 \sin^2(\beta h)$$

$$= 1 - 2p(1-p)(1 - \cos(\beta h)) \geq 0, \quad -\pi < \beta h \leq \pi$$

So, for $p = 1$, $|\xi|^2 = 1$ and there are no dissipation

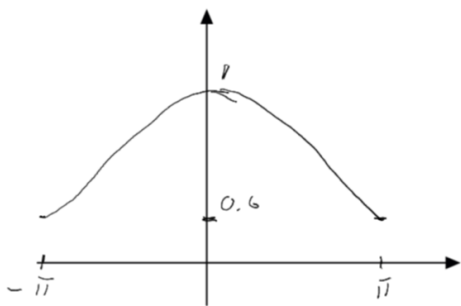
$p > 1$ or $p < 0$: $|\xi|^2 > 0$ for some βh , and the scheme is unstable. This coincides with the CFL-condition.

$0 < p < 1$: In this case, $|\xi| \leq 1$ and the scheme is stable. ξ depends on βh , so the method is dissipative. It can be shown that

$$|\xi| = 1 - \frac{1}{2}p(p-1)(\beta h)^2 + O((\beta h)^4)$$

thus the method is dissipative of order 2.

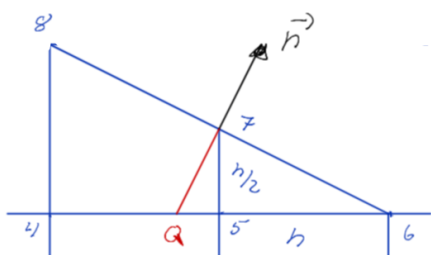
For $p = 0.8$, $\xi = \sqrt{0.68 + 0.3 \cdot \cos(\beta h)}$, is given by



and high frequency modes will be damped faster than slow modes. Consequently, the the solution will be smoothed out over time.



Problem 5:



We notice that the unit vector \vec{n} is

$$\vec{n} = \frac{1}{\sqrt{5}} [1, 2].$$

Thus

$$\partial_n u = \nabla u \cdot \vec{n} = \frac{1}{\sqrt{5}} (u_x + 2u_y).$$

We search for an approximation to $\partial_n u$ of the form

$$a_5 u_5 + a_6 u_6 + a_7 u_7 + a_8 u_8 \approx \frac{1}{\sqrt{5}} (u_x + 2u_y) \text{ at point 7.}$$

Let point 7 be given by (x_7, y_7) and do a Taylor expansion around this point:

$$\begin{aligned} & a_5 u(x_7, y_7 - \frac{1}{2}h) + a_6 u(x_7 + h, y_7 - \frac{1}{2}h) + a_7 u(x_7, y_7) + a_8 u(x_7 - h, y_7 + \frac{1}{2}h) \\ &= a_5 (u - \frac{1}{2}h u_y + \frac{1}{8}h^2 u_{yy} + \dots) \\ &+ a_6 (u + h u_x - \frac{1}{2}h u_y + \frac{1}{2}h^2 u_{xx} - \frac{1}{2}h^2 u_{xy} + \frac{1}{8}h^2 u_{yy} + \dots) \\ &+ a_7 u \\ &+ a_8 (u - h u_x + \frac{1}{2}h^2 u_y + \frac{1}{2}h^2 u_{xx} - \frac{1}{2}h^2 u_{xy} + \frac{1}{8}h^2 u_{yy} + \dots) \\ &= (a_5 + a_6 + a_7 + a_8) u + h [(a_6 - a_8) u_x + (-\frac{1}{2}a_5 - \frac{1}{2}a_6 + \frac{1}{2}a_8) u_y] \\ &+ h^2 [(\frac{1}{2}a_6 + \frac{1}{2}a_8) u_{xx} + (-\frac{1}{2}a_6 - \frac{1}{2}a_8) u_{xy} + (\frac{1}{8}a_5 + \frac{1}{8}a_6 + \frac{1}{8}a_8) u_{yy}] \\ &+ \mathcal{O}(h^3) \end{aligned}$$

So, for this to be a consistent approximation, we require

$$u: \quad a_5 + a_6 + a_7 + a_8 = 0$$

$$u_x: \quad h(a_6 - a_8) = \frac{1}{\sqrt{5}} \quad , \quad u_y: \quad -\frac{1}{2}(a_5 + a_6 - a_8) = \frac{2}{\sqrt{5}}$$

which leaves us with one extra free variable, choose for inst.

$$a_5 = -\frac{\sqrt{5}}{h}, \quad a_8 = a_6 - \frac{1}{\sqrt{5}h}, \quad a_7 = \frac{6}{\sqrt{5}h} - 2a_6$$

Taking a look at the h^2 -terms, we observe that the u_{xx} and the u_{yy} terms becomes 0 if $a_6 = -a_8$. Using this, we end up with:

$$a_5 = -\frac{\sqrt{5}}{h}, \quad a_6 = \frac{2}{\sqrt{5}h}, \quad a_7 = \frac{\sqrt{5}}{h}, \quad a_8 = -\frac{2}{\sqrt{5}h}$$

and the scheme becomes:

$$\frac{1}{\sqrt{5}h} (-5U_5 + 2U_6 + 5U_7 - 2U_8) = g(x_7, y_7).$$

The local truncation error is now

$$\tau_7 = \frac{1}{8} (a_5 + a_6 + a_8) u_{yy} + \mathcal{O}(h^2) = - \frac{\sqrt{5}}{8} h u_{yy} + \mathcal{O}(h^2)$$

Problem 6.

$$-(a(x) \cdot u_x)_x = 1 \quad 0 < x < 1$$

$$u_x(0) = 1, \quad u(1) = 0.$$

a) Multiply by a function v , use partial integration to get

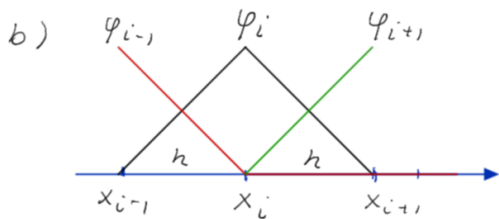
$$\int_0^1 a u_x v_x dx - \int_0^1 u_x v = \int_0^1 v dx$$

For the integrals to exist, $u, v \in H^1(\Omega)$, $\Omega = (0, 1)$. We know that $u_x(0) = 1$. To get rid of the boundary term at $x=1$, we require $v(0) = 0$ (and we know that $u(0) = 0$).

$$\text{Let } V = H_n^1(\Omega) = \{v \in H^1(\Omega), v(1) = 0\} \quad (\Omega = (0, 1)).$$

The weak formulation is now:

$$\text{Find } u \in H_n^1 \text{ s.t. } \int_0^1 a u_x v_x dx = \int_0^1 v dx - v(1), \quad \forall v \in H_n^1(\Omega)$$



The linear nodal basis functions are described in the figure.

$$\varphi_i' = \begin{cases} 1/h & x_{i-1} < x < x_i \\ -1/h & x_i \leq x < x_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

The elements of the stiffness matrix is

$$A_{ij} = \int_0^1 (x + \frac{1}{2}) \varphi_i' \varphi_j' dx, \quad i = 0, \dots, M-1$$

The matrix is tridiagonal, with elements: $i = 1, \dots, M-1$

$$A_{ii} = \frac{1}{h^2} \int_{x_{i-1}}^{x_{i+1}} (x + \frac{1}{2}) dx = \frac{1}{h} (2x_i + 1)$$

$$A_{i,i+1} = -\frac{1}{h^2} \int_{x_i}^{x_{i+1}} (x + \frac{1}{2}) dx = -\frac{1}{h} (x_i + \frac{1}{2}(1+h))$$

$$A_{i,i-1} = -\frac{1}{h^2} \int_{x_{i-1}}^{x_i} (x + \frac{1}{2}) dx = -\frac{1}{h} (x_i + \frac{1}{2}(1-h))$$

For the boundary function φ_0 , we get.

$$A_{00} = \frac{1}{h^2} \int_0^h (x + \frac{1}{2}) dx = \frac{1}{2h} (1+h)$$

$$A_{01} = -\frac{1}{h^2} \int_0^h (x + \frac{1}{2}) dx = -\frac{1}{2h} (1+h)$$

For the load vector \vec{b} , all elements
but the first is

$$b_i = \int_{x_{i-1}}^{x_{i+1}} 1 \cdot \varphi_i(x) dx = 2 \cdot \frac{1}{2} h = h$$

For the first one we get

$$b_0 = \int_0^h 1 \cdot \varphi_0(x) dx + \varphi_0(1) = \frac{h}{2} + 1$$