NTNU
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Department of Mathematical Sciences

Examination paper for
TMA4212 Numerical Solution of Differential Equations by Difference Methods

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Permitted examination support material: Code C:
Approved simple pocket calculator.
The two notes by Brynjulf Owren on finite differences and Charles Curry on finite elements.
Rottmann matematisk formelsamling.

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Informasjon om trykking av eksamensoppgave	
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Date Signature

All answers must be properly argued for.

Problem 1 Consider the equation

$$\begin{aligned} u_t &= a u_{xx} - u, \qquad 0 < x < 1, a > 0 \\ u(0,t) &= u(1,t) = 0, \\ u(x,0) &= f(x) \end{aligned}$$

a) Set up a finite difference scheme for this problem, using a central difference with stepsize h in the x-direction, and a backward difference with stepsize k in the t-direction. Let $h = 1/M$.

Find the local truncation error of the scheme.

b) Prove that the scheme is convergent for all values of $r = ak/h^2$.

Problem 2 Consider the hyperbolic differential equation

$$\begin{aligned} u_t + \frac{1}{(1+t)^2} u_x &= 0, \quad 0 < x < 1, \quad t > 0 \\ u(x,0) &= f(x) \end{aligned}$$

a) Find the characteristic $x(t)$ of the above equations through some point (x^*, t^*). Indicate how the characteristics look like by making a sketch.

On which of the boundaries $x = 0$ or $x = 1$ should there be a boundary condition?

b) Assume that the problem is solved by an explicit difference formula of the form

$$U_{m+1}^{n+1} = a_{m+1}^n U_{m+1}^n + a_m^n U_m^n + a_{m-1}^n U_{m-1}^n,$$

using a uniform grid with stepsize h in the x-direction and k in the t-direction. Set $h = 0.25$. What restriction is required on k for the method to satisfy the CFL condition for all $t > 0$?

Problem 3 Given the equation

$$\begin{aligned} u_t + v(u)u_x &= (a(u)u_x)_x, \qquad\qquad 0 < x < 1 \\ u(0,t) &= 0, \qquad u_x(1,t) + u(1,t) = 0, \\ u(x,0) &= f(x) \end{aligned}$$

where the given functions $v(u)$ and $a(u)$ are both positive.

Suggest a semi-discretization of the problem.

Choose an appropriate method for solving the resulting system of ordinary differential equations, and write down the complete scheme.

Justify your choices.

Problem 4 Consider the transport equation

$$u_t + au_x = 0, \qquad -\infty < x < \infty$$

Investigate the dissipation properties of the upwind method

$$U_m^{n+1} = U_m^n - p(U_m^n - U_{m-1}^n),$$

where $p = ak/h$.

Given the intial conditions

$$u(x,0) = f(x) = \begin{cases} 1, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

We know that the exact solution of the transport equation is $u(x,t) = f(x - at)$, that is, the inital rectangle moves along x with speed a.

Assuming that $p = 0.8$, how would you expect the initial profile to be deformed over time? Make a sketch.
What if $p = 1$?

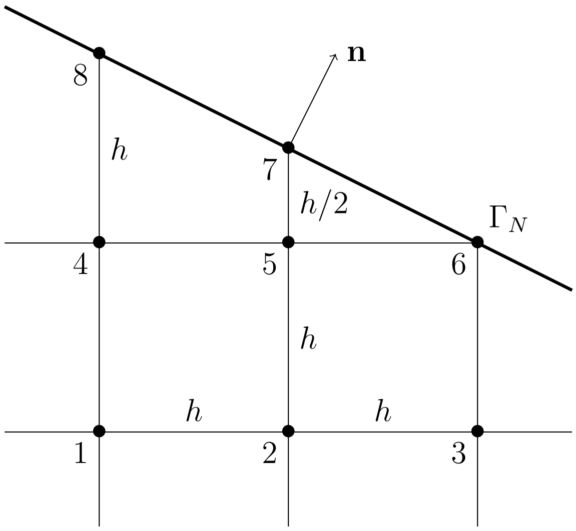
Problem 5
We want to solve the Poisson equation

$$\Delta u = u_{xx} + u_{yy} = f(x, y).$$

on a domain Ω, using the 5-point formula on a regular grid with stepsize h. Part of the domain, including the grid, is depicted in the figure below. At the boundary Γ_N the following boundary condition holds

$$\frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n} = g(x, y)$$

for a given function $g(x, y)$.



Find an approximation of the boundary condition in node 7 using the solutions in the nodes 5,6,7 and 8. Find the local truncation error of the approximation, and give its order.

Problem 6 Given the equation

$$-(a(x)u_x)_x = 1, \qquad 0 < x < 1,$$
$$u_x(0) = 1, \quad u(1) = 0,$$

where $a(x) > 0$ is some known, positive function of x.

a) The weak formulation of the problem is given by

Find $u \in V$ such that $a(u, v) = F(v)$ for all $v \in V$.

Find the function space V, the bilinear form $a(u, v)$ and the functional $F(v)$ in this case.

b) This problem is solved by the linear finite element method on a uniform grid $x_i = ih$, $i = 0, \dots, M$ with $h = 1/M$, using linear nodal basis functions. Show that this results in a linear system of equations:

$$A\mathbf{u} = \mathbf{b}.$$

Let $a(x) = 1/2 + x$ and find the stiffness matrix A and the load vector $\mathbf{b}$ for a general value of M.