Problem 1 Let $a \geq 0$, $b, c \in \mathbb{R}$, and $bc \geq 0$. Then the eigenvalues of

$$A_{a,b,c} := \begin{pmatrix} a & b & 0 \\ c & a & b \\ 0 & c & a \end{pmatrix} \qquad \text{are} \qquad \lambda_1 = a, \quad \lambda_2 = a - \sqrt{2bc}, \quad \lambda_3 = a + \sqrt{2bc}.$$

Assume $A := A_{2,b,c}$ $(a = 2)$ is symmetric positive definite.

Which values of b and c make A symmetric positive definite?

Show that the Jacobi iteration for $A\vec{x} = \vec{b}$ converges for any fixed $\vec{b} \in \mathbb{R}^3$.

Problem 2 We want to solve the initial-boundary value problem

$$(1) \qquad \begin{cases} u_t - u_{xx} + xu_x = 1 & \text{in} \quad (0,1) \times (0,T), \\ u(0,t) = 1, \quad u(1,t) = e & \text{in} \quad (0,T), \\ u(x,0) = e^x & \text{in} \quad (0,1), \end{cases}$$

by finite difference methods on the grid

$$\Big\{(x_m, t_n) = (mh, nk) : m = 0, \dots, M, \quad n = 0, \dots, N\Big\},$$

where $h = \frac{1}{M}$ and $k = \frac{T}{N}$. Let $U_m^n \approx u(x_m, t_n)$ be the approximate solution.

a) Consider a consistent discretisation of (1) by backward differences in time and central differences in space. With $\vec{U}^n = [U_1^n, \dots, U_{M-1}^n]^T$, we get that

$$A_h \vec{U}^{n+1} = \vec{U}^n + \vec{F}^n, \quad n = 1, \dots, N-1; \qquad \vec{U}^0 = \vec{G}.$$

Determine A_h, $\vec{F}^n$, and $\vec{G}$.

b) Consider a consistent discretisation of (1) by forward differences in time and central differences in space. Write the scheme in the form

$$\alpha_{m,m}^{n+1} U_m^{n+1} - \sum_{\ell} \alpha_{m,\ell}^n U_\ell^n = F_m^n, \quad m = 1, \dots, M-1,$$

and find the CFL condition that makes the scheme monotone.

Problem 3 Consider the boundary value problem

$$(2)\qquad \begin{cases} -Lu(x) := -(\frac{d^2}{dx^2} - x\frac{d}{dx} - 1)u(x) = 1 \qquad \text{in} \qquad (0,1), \\ u(0) = 1, \quad u(1) = 2, \end{cases}$$

On the grid $\{x_m = mh : m = 0, \dots, M\}$, $h = \frac{1}{M}$, we discretise this problem by

$$(3)\qquad \begin{cases} -L_h U_m := -(\frac{\delta_h^2}{h^2} - x_m\frac{\nabla_h}{h} - 1)U_m = 1, \quad m = 1, \dots, M-1, \\ U_0 = 1, \quad U_M = 2. \end{cases}$$

where δ_h^2 and ∇_h are central and backward differences, and $U_m \approx u(x_m)$.

a) Compute the local truncation error for the scheme (3),

$$\tau_m = -(L - L_h)u(x_m), \qquad m = 1, \dots, M-1,$$

when $u \in C^4([0,1])$.

b) Show that the scheme (3) is stable with respect to the right hand side:

$$-L_h V_m = f_m, \quad m = 1, \dots, M-1; \qquad V_0 = 0, \quad V_M = 0;$$
$$\Downarrow$$
$$\max_m |V_m| \le \max_m |f_m|.$$

Hint: You may use without proof that the scheme is monotone and (therefore) the discrete maximum principle holds. Note that $-L_h 1 = 1$.

c) Show the error bound

$$\max_m |u(x_m) - U_m| \le \frac{1}{2}\|u''\|_{L^\infty(0,1)}h + \frac{1}{12}\|u^{(4)}\|_{L^\infty(0,1)}h^2.$$

Problem 4 Consider the boundary value problem for the 1d Poission equation,

$$-\frac{d}{dx}\Big(\sin x\,\frac{du}{dx}\Big) = \cos x \qquad \text{in} \qquad \Omega := (\tfrac{\pi}{6}, \tfrac{\pi}{2}), \tag{4}$$

$$u(\tfrac{\pi}{6}) = 1 \quad \text{and} \quad u(\tfrac{\pi}{2}) = 0. \tag{5}$$

a) The weak/variational formulation for problem (4)–(5) is to find $u \in X$ such that the boundary condition (5) holds and

$$a(u, v) = F(v) \qquad \text{for all} \qquad v \in V.$$

Find the correct choices for X, $V (\subset X)$, a, and F here, and show that $a(\cdot, \cdot)$ is a continuous (and) bilinear form on $X \times X$.

Approximate problem (4)–(5) by the $\mathbb{P}_1$ finite element method on the triangulation:

$$K_1 = (\tfrac{\pi}{6}, \tfrac{\pi}{4}), \qquad K_2 = (\tfrac{\pi}{4}, \tfrac{\pi}{3}), \qquad K_3 = (\tfrac{\pi}{3}, \tfrac{\pi}{2}).$$

Let the finite element solution be $u_h(x) = \sum_{i=0}^{3} U_i \phi_i(x) \in X_h^1(\frac{\pi}{6}, \frac{\pi}{2})$.

b) Explain why the finite element method is equivalent to a linear system of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ * & * & c & 0 \\ 0 & c & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix}$$

and compute c.