TMA4212
Numerical solution of differential equation by difference methods

Semester project

This years semester project is about solving a PDE efficiently in Matlab using a difference method. You shall solve *one* of the 4 PDE's listed below. We encourage you to write efficient programs since there will be a competition where groups solving the same equation will be challenged to solve a given problem. The group with the shortest running time (for each PDE) will be rewarded.

Practical information

This is what you do:

1. Form a group of 3 students, and choose a contact person.

2. Send an e-mail to teaching assistant Jens Birkevold, before **Tuesday 28th February 12:15 PM**, with the name of all the group members (including e-mail adresses), who the contact person is, and make a list of the PDE's where you rank them according to which you want to work with the most. We will try to fulfill everyones wishes, but we do not want to have too many groups working with the same equation.

3. Jens Birkevold will give you a group number and tell you which PDE you will be working with. A time for evaluation of the Matlab-code will be given later.

4. Time for the guidance lessons will be:

Monday 05.03	14.15 - 15.00
Wednesday 07.03	10.15 - 11.00
Monday 12.03	14.15 - 15.00
Wednesday 14.03	10.15 - 11.00
Monday 19.03	14.15 - 15.00
Wednesday 11.04	10.15 - 11.00
Monday 16.04	14.15 - 15.00
Wednesday 18.04	10.15 - 11.00

   All the lessons will be in *Nullrommet* (Room 380A in Sentralbygg 2). If there are many questions the lessons will last another hour.
   It is preferred that you use these lessons to ask questions, but you can ask the lecturer after the lectures or make an appointment.

5. The report should not exceed 15 pages (this is not an absolute limit, but if you write more it should be *really* important). You should also list the Matlab-code in an appendix at the end of the report (this is not a part of the 15 pages). On the front page of the report you must put the names of all the group members and the group number. The report should be sent to Jens Birkevold before the deadline; **27th April**.

6. There will be a numerical competition where groups solving the same PDE will be challenged to solve the PDE with given data (initial, boundary, etc.), and the teaching assistant will time the Matlab-program. The winner group will be given up to 5 extra points (out of the maximum of 40 points for the semester project, but the total score can not exceed 40).

7. On the 24th of april there will be a poster presentation of the projects on the 13th floor of SB II. Each group must make a stand with a poster (see the homepage for help to create a poster), and answer questions from members of the institute passing by. The lecturer and teaching assistant will also listen to the presentations, and this counts approximately 1/3 of the score on the semester project.

The exercise

How much information there is given about each equation can vary. The main idea has been to give a very short description, sometimes just the equation, so that is up to the group to find the necessary information. Some keywords are given for each equation; use these to search for information.
The exercise can be divided into multiple stages.

1. Get familiar with the equation:

   - Which type of boundary and initial conditions should be used (if this is not given)?
   - Where does the equation come from? What does it describe?
   - Is there any analytical solutions?

   This will be a nice introduction in the report.

2. Discretize the equation using difference methods, but you could also try other methods. Analyze the truncation error, and if you can, say something about the stability for the numerical method.

3. Implement the method in Matlab. This should be described in detail; how the matrices look like, how the boundary conditions is implemented, etc. There should also be a verification of the method, for instance there should be an experiment showing the expected order of convergence. If there is one, the exact solution can be used to do this. To compare a plot of the exact solution and the numerical solution, and say that they are *almost equal* is not enough.

4. At the end of the report there should be a presentation of the results. Try to make examples that illustrate what the equation describes. This is an important part; try to make use of some of the graphical utilities Matlab offers.

Evaluation

The semester project counts 40 % of the total grade in this course. (The final exam counts 60 %.) You can take the final exam without handing in the semester project, but then you can only achieve a **C** at the best. Even if you get a top score on the semester project, you need 35-40 points out of 100 on the final exam to pass the course.

The score on the semester project depends on three criteria:

1. The lecturers evaluation of the report.

2. The lecturers evaluation of the poster presentation.

3. The lecturers and the teaching assistants evaluation of the demonstration of the Matlab-code (running time and memory usage are important factors).

The PDE's you can choose from:

PDE-1: Porous medium equation

We will look at the equation

$$u_t = (u^m u_x)_x$$

as a initial value problem, or as a two sided boundary value problem; for instance $u(0,t) = g_0(t)$, $u(1,t) = g_1(t)$, $u(x,0) = f(x)$. Find relevant values for m in the literature.

Keywords: Porous medium equation, "Elliot, Herrero, King and Ockendon", Barenblatt solution

PDE-2: Burgers' equation

We will look at the equation

$$u_t + uu_x = \varepsilon u_{xx}$$

as a two sided boundary value problem, for instance $u(0,t) = g_0(t)$, $u(1,t) = g_1(t)$, $u(x,0) = f(x)$.
The idea is to let ε become very small, and study the numerical complication this causes. Is there a cure?

Keywords: Burgers' equation, conservation laws, viscosity solution, moving mesh methods

PDE-3: Biharmonic equation

We will look at the inhomogene case

$$\Delta^2 u = f(x), \qquad \mathbf{x} \in \Omega$$

where $\Omega \subset \mathbb{R}^d$, and $\Delta = \nabla^2$ is the Laplace operator. The solution must satsify these boundary conditions:

$$u(\mathbf{x}) = \Delta u(\mathbf{x}) = 0, \qquad \mathbf{x} \in \partial\Omega.$$

You only need to look at the case where $d = 2$, that is $\mathbf{x} = (x, y)$, and let Ω be a rectangel. In cartesian coordinates we have

$$\Delta^2 u = u_{xxxx} + 2u_{xxyy} + u_{yyyy}.$$

If you have time, you could let Ω be a disc with radius 1, and look at the equation in polar coordinates.

Keywords: Biharmonic equation, thin plate, biharmonic operator, Laplacian

PDE-4: 2D Stationary convection diffusion equation

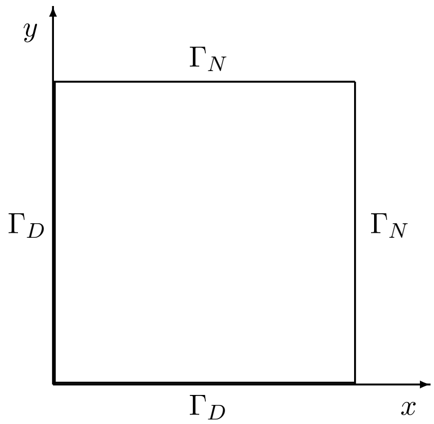
Also known as *Stationary advection diffusion equation*

Given a divergence-free vector field $\mathbf{V}$ (that is $\nabla \cdot \mathbf{V} = 0$) on the square $0 \leq x \leq 1$, $0 \leq y \leq 1$; look at the equation

$$\nu \nabla^2 u + \mathbf{V} \cdot \nabla u = f(x, y).$$

Let the boundary $\Gamma = \Gamma_D \cup \Gamma_N$ be as shown in the figure.

$$\begin{aligned}
\Gamma_D &= \{(x,0) : 0 \leq x \leq 1\} \cup \{(0,y) : 0 \leq y \leq 1\} \\
\Gamma_N &= \{(x,1) : 0 \leq x \leq 1\} \cup \{(1,y) : 0 \leq y \leq 1\}
\end{aligned}$$



Boundary conditions:

$$\begin{aligned} u(x,y) &= u^*(x,y), \qquad \text{when } (x,y) \in \Gamma_D \\ \nabla u \cdot \vec{n} &= 0, \qquad \text{when } (x,y) \in \Gamma_N \end{aligned}$$

where u^* is a function of x and y, and $\vec{n}$ is an outward unit normal vector.

From Matematikk 2 (or a similar course) you should have an idea on have to create a divergence-free vector field, but feel free to ask.

Keywords: Stationary, advection/convection, diffusion.