Department of Mathematical Sciences

Examination paper for
MA8109 Stochastic Prosesses in Engineering Systems

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Examination time (from–to): 15:00–19:00
Permitted examination support material: C: Approved calculator, list of fomulas that comes with the exam.

Other information:
There is a list of usefull formulas at the end of this exam, read it *before* you start to work.

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Checked by:

Date Signature

Problem 1

Let B_t be a Brownian motion in $\mathbb{R}^1$, show that $\tilde{B}_t = B_{t+1} - B_1$ is another Brownian motion in $\mathbb{R}^1$.

Problem 2 Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of $[0, T]$ and

$$\phi_N(t, \omega) = \sum_{j=0}^{N} B_{t_j}^2(\omega)\chi_{[t_j, t_{j+1})}(t),$$

where $B_t(\omega)$ is a Brownian motion in $\mathbb{R}^1$ and the function $\chi_A(t)$ is 1 when $t \in A$ and 0 otherwise.

a) Give the definitions of the two Ito integrals

(i) $\int_0^T \phi_N dB_s$,

(ii) $\int_0^T B_s^2 dB_s$.

Hint: Do not compute these integrals.

b) Find the expectation and variance of

(i) $X_t = \int_0^t s dB_s$,

(ii) $Y = \int_0^T B_s^2 dB_s$.

Hint: The formulas at the end of the exam may be usefull.

Let $\tilde{\phi}$ be the elementary function defined by

$$\tilde{\phi}(t, \omega) = \begin{cases} 0, & t \in [0, 1), \\ B_1^2(\omega), & t \geq 1, \end{cases}$$

where B_1 is the Brownian motion at $t = 1$.

c) Let $\{\mathcal{F}_t\}_t$ be the filtration generated by the Brownian motion B_t and define

$$X_t(\omega) = \int_0^t \tilde{\phi}(s, \omega) dB_s(\omega).$$

Show by a direct argument that X_t is a martingale w.r.t. the filtration $\{\mathcal{F}_t\}_t$.

Hint: The formulas at the end of the exam may be usefull.

Problem 3

Physicists have suggested the following (scaled) model for the position of a pollen particle suspended in a liquid that experiences periodic external forcing,

$$\ddot{X} + \dot{X} = W + \sin t,$$

where W is Gaussian white noise.

a) Write down the Ito interpretation of this equation.

b) Solve the Ito stochastic differential equation obtained in a).
Hint: Use an integrating factor.

Problem 4 Consider the following stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \qquad X_0 = Z, \tag{1}$$

where b and σ are continuous functions and Z is a random variable.

Give sufficient conditions on b, σ, and Z to have existence and uniqueness of a strong solution of (1).

Define what it means for X_t to be a strong solution of (1).

Problem 5

Let $f \in C_c^2(\mathbb{R}^2)$ and let $u \in C^{1,2}(\mathbb{R} \times \mathbb{R}^2)$ be the solution of

$$\begin{aligned} \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \qquad &\text{for} \qquad (x, y) \in \mathbb{R}^2,\ t > 0, \\ u(0, x, y) = f(x, y) \qquad &\text{for} \qquad (x, y) \in \mathbb{R}^2. \end{aligned}$$

Use an Ito diffusion to find an integral formula for $u(t, x, y)$.

Hint: The formulas at the end of the exam may be usefull.

Problem 6 Let $\tau_{U_R}^x$ be the exit-time from $U_R = \{x : |x| < R\}$ of the following one-dimensional Ito diffusion:

$$X_t^x = x + \int_0^t (2 + \sin X_s^x) dB_s.$$

Prove that for all $x \in U_R$ and $R > 0$,

$$E(\tau_{U_R}^x) < \infty.$$

Hint: Try the "usual argument". Formulas at the end of the exam may be usefull.

Problem 7 Let X_t and X_t^ϵ be (strong) solutions of the one-dimensional (stochastic) differential equations

$$dX_t = b(X_t)dt, \qquad X_0 = x_0,$$
$$dX_t^\epsilon = b(X_t^\epsilon)dt + \epsilon\, dB_t, \qquad X_0^\epsilon = x_0,$$

where $x_0, \epsilon \in \mathbb{R}$, and for all $x, y \in \mathbb{R}$,

$$|b(x) - b(y)| \leq L|x - y|.$$

Prove that there is a $C > 0$ such that

$$E\left(|X_t - X_t^\epsilon|^2\right) \leq Ce^{Ct}\epsilon^2.$$

Hint: The formulas at the end of the exam may be usefull.

List of useful formulae

Note: The list does not state the requirements for the formulae to be valid.

1D Gaussian variable: $X \in \mathcal{N}(\mu, \sigma^2)$;

(i) $\mathsf{E}(X-\mu)^4 = 3\sigma^4$,

(ii) $\Phi_X(u) = e^{i\mu - \frac{1}{2}\sigma^2 u^2}$,

(iii) $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{(x-\mu)^2}{2\sigma}}$.

Conditional Expectations:

(i) If Y is $\mathcal{H}$-measurable, then $\mathsf{E}(YX|\mathcal{H}) = Y\mathsf{E}(X|\mathcal{H})$.

(ii) If X is independent of $\mathcal{H}$, then $\mathsf{E}(X|\mathcal{H}) = \mathsf{E}(X)$.

(iii) If $\mathcal{G} \subset \mathcal{H}$, then $E(E(X|\mathcal{H})|\mathcal{G}) = E(X|\mathcal{G})$.

Itô Isometry: $\mathsf{E}\left|\int_0^T f(t,\omega)\, dB_t(\omega)\right|^2 = \int_0^T \mathsf{E}\,|f(t,\omega)|^2\, dt = \|f\|^2_{L^2(\Omega\times[0,T])}$

2D Itô Formula: The "Rules" and

$$dg(t, X_t, Y_t) = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial x}dX_t + \frac{\partial g}{\partial y}dY_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dX_t)^2 + \frac{\partial^2 g}{\partial x \partial y}dX_t dY_t + \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(dY_t)^2.$$

The Generator for $dX_t = b(X_t)\, dt + \sigma(X_t)\, dB_t$:

$$A(f)(x) = \sum_{i=1}^n b_i(x)\frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^n \left(\sigma(x)\sigma(x)^T\right)_{i,j}\frac{\partial^2 f}{\partial x_i \partial x_j}(x).$$

Dynkin's formula: $Ef(X^x_\tau) = f(x) + E\left(\int_o^\tau Af(X^x_s)ds\right)$.

Grönwall's inequality: If $v(t) \le C + A\int_0^t v(s)\, ds \;\; \ldots$, then $v(t) \le Ce^{At}$.

Jensen's inequality: $\left(\int_0^t v(s)\, ds\right)^2 \le t\int_0^t v^2(s)\, ds$.