SUPERCALORIC FUNCTIONS FOR THE POROUS MEDIUM EQUATION

JUHA KINNUNEN, PEKKA LEHTELÄ, PETER LINDQVIST, AND MIKKO PARVIAINEN

Abstract. In the slow diffusion case unbounded supersolutions of the porous medium equation are of two totally different types, depending on whether the pressure is locally integrable or not. This criterion and its consequences are discussed.

1. Introduction

The porous medium equation

$$u_t - \Delta(u^m) = 0, \quad m > 1, \tag{1.1}$$

has a well developed theory for its solutions treated, for example, in the monographs [4], [8], [14] and [15]. Here $u = u(x,t)$ is a non-negative function on $\Omega_T = \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$. We are interested in supersolutions of (1.1) in the slow diffusion case $m > 1$. A supersolution should satisfy the inequality $u_t - \Delta(u^m) \geq 0$, but this is a delicate issue. For a function $u : \Omega_T \to [0,\infty]$, we consider the two definitions below:

- (Weak supersolution) We say that u is a weak supersolution, if $u^m \in L^2_{\mathrm{loc}}(0,T; W^{1,2}_{\mathrm{loc}}(\Omega))$ and
$$\iint_{\Omega_T} (-u\varphi_t + \nabla(u^m)\cdot\nabla\varphi)\, dx\, dt \geq 0 \tag{1.2}$$
for every non-negative test function $\varphi \in C_0^\infty(\Omega_T)$.
- (m-supercaloric function) We say that u is m-supercaloric, if
  - (i) u is lower semicontinuous,
  - (ii) u is finite in a dense subset of Ω_T and
  - (iii) u obeys the comparison principle with respect to solutions in every subdomain $D_{t_1,t_2} = D \times (t_1,t_2)$, $D_{t_1,t_2} \Subset \Omega_T$: if $h \in C(\overline{D_{t_1,t_2}})$ is a weak solution of (1.1) in D_{t_1,t_2} and $u \geq h$ on the parabolic boundary $\partial_p D_{t_1,t_2}$, then $u \geq h$ in D_{t_1,t_2}.

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In a similar manner, we may also consider solutions defined, for example, in $\Omega \times (-\infty, \infty)$ or in $\mathbb{R}^n \times \mathbb{R}$. Since our results are local, we may restrict ourselves to space-time cylinders in $\mathbb{R}^{n+1}$. The case $m = 1$ gives supercaloric functions for the heat equation.

Several remarks are appropriate. First, weak supersolutions obey the comparison principle, see [8], [14] and [15], and by [1] they are lower semicontinuous, after a possible redefinition on a set of $(n+1)$-dimensional Lebesgue measure zero. Thus every weak supersolution is an m-supercaloric function. Second, by [9] every bounded m-supercaloric function is a weak supersolution. In particular, they belong to the natural Sobolev space. Thus if we only consider bounded functions then the classes of m-supercaloric functions and weak supersolutions coincide. The advantage of weak supersolutions is that they satisfy expedient Caccioppoli and Harnack estimates. The class of non-negative m-supercaloric functions is even more flexible. For example, it is closed under increasing convergence, if the limit function is finite on a dense set. It is closed under taking minimum of finitely many m-supercaloric functions. Moreover, a non-negative m-supercaloric function may be redefined to be zero until a given moment of time. These properties will be useful for us later.

The examples below show that m-supercaloric functions are by no means innocent. Let $q_1, q_2, \ldots$ be the points with rational coordinates in $\mathbb{R}^n$. The stationary function

$$u(x,t) = \left(\sum_{i=1}^{\infty} \frac{a_i}{|x - q_i|^{n-2}} \right)^{\frac{1}{m}}, \quad n \geq 3,$$

is a weak supersolution in $\mathbb{R}^{n+1}$, provided that the coefficients $a_i > 0$ are chosen properly. Now $u(q_i, t) \equiv \infty$, $i = 1, 2, \ldots$. Nevertheless, $u^m \in L^2_{\text{loc}}(\mathbb{R}^n; W^{1,2}_{\text{loc}}(\mathbb{R}^n))$.

In general, the class of m-supercaloric functions is wider: it contains important functions that fail to be weak supersolutions. The two important examples for us are:

- The Barenblatt solution

$$\mathcal{B}(x,t) = \begin{cases} t^{-\lambda} \left(C - \dfrac{\lambda(m-1)}{2mn} \dfrac{|x|^2}{t^{\frac{2\lambda}{n}}} \right)_+^{\frac{1}{m-1}}, & t > 0, \\ 0, \quad t \leq 0, \end{cases} \tag{1.3}$$

where $m > 1$, $\lambda = \frac{n}{n(m-1)+2}$ and $C > 0$. This function is m-supercaloric in $\mathbb{R}^{n+1}$, but it is not a weak supersolution in any domain that contains the origin, since

$$\int_{-1}^{1} \int_{|x|<r} |\nabla(\mathcal{B}(x,t)^m)|^2 \, dx \, dt = \infty.$$

However, it is a weak solution in $\mathbb{R}^{n+1}\backslash\{0\}$. Moreover, $\mathcal{B} \in L^q_{\text{loc}}(\mathbb{R}^n \times \mathbb{R})$ whenever $q < m + \frac{2}{n}$, the weak gradient exists and $\nabla(\mathcal{B}^m) \in L^q_{\text{loc}}(\mathbb{R}^n \times \mathbb{R})$ for every $q < 1 + \frac{1}{1+mn}$, see [9]. Furthermore,

$$\mathcal{B}_t - \Delta(\mathcal{B}^m) = C\delta,$$

where δ is Dirac's delta and $C = C(m, n) > 0$.

- The friendly giant

$$V(x,t) = \begin{cases} \dfrac{U(x)}{(t-t_0)^{\frac{1}{m-1}}}, & t > t_0, \\ 0, \quad t \le t_0, \end{cases} \tag{1.4}$$

where $x \in \Omega$, $\Omega \subset \mathbb{R}^n$ is a bounded domain. Here $U > 0$ satisfies the auxiliary elliptic equation

$$\Delta(U^m) + \frac{1}{m-1}U = 0 \tag{1.5}$$

with the zero boundary values in Ω. This function is m-supercaloric, but it is not a weak supersolution in $\Omega \times \mathbb{R}$. However, V is a weak solution in $\Omega \times (t_0, \infty)$. A characteristic feature of the friendly giant is a total blow-up at a time slice

$$\lim_{\substack{(y,t)\to(x,t_0) \\ t>t_0}} V(y,t) = \infty \quad \text{for every} \quad x \in \Omega.$$

See [14, p. 111–114].

Notice that $\mathcal{B}^{m-1} \in L^1_{\text{loc}}(\mathbb{R}^{n+1})$ while $V^{m-1} \notin L^1_{\text{loc}}(\Omega_T)$. This summability for the pressure is decisive. Unbounded m-supercaloric functions are divided into two mutually exclusive classes $\mathfrak{B}$ and $\mathfrak{M}$ depending on whether the pressure $\frac{u^{m-1}}{m-1}$ is locally integrable or not. The following results were outlined in [11] and our aim now is to provide complete proofs. Let $m > 1$. For an m-supercaloric function $u : \Omega_T \to [0, \infty]$ the following conditions are equivalent:

- **Class $\mathfrak{B}$**
  - (i) $u \in L^{m-1}_{\text{loc}}(\Omega_T)$,
  - (ii) $u \in L^q_{\text{loc}}(\Omega_T)$ whenever $q < m + \frac{2}{n}$,
  - (iii) the Sobolev gradient $\nabla(u^m)$ exists and belongs to $L^q_{\text{loc}}(\Omega_T)$ whenever $q < 1 + \frac{1}{1+mn}$,

  The proof is given in Section 3. Notice the gap $[m-1, m+\frac{2}{n})$. Furthermore, functions of class $\mathcal{B}$ satisfy a measure equation

$$u_t - \Delta(u^m) = \mu,$$

  where μ is a non-negative a Radon measure on $\mathbb{R}^{n+1}$.
- **Class $\mathfrak{M}$**
  - (i) $u \notin L^{m-1}_{\text{loc}}(\Omega_T)$,

(ii) there exists a time $t_0 \in (0,T)$ such that

$$\lim_{\substack{(y,t)\to(x,t_0)\\ t>t_0}} u(y,t) = \infty \quad \text{for every} \quad x \in \Omega.$$

The proof is given in Section 4. Functions of class $\mathfrak{M}$ have very few, if any good properties. In particular, they do not induce a Radon measure.

Finally we also study the infinity sets, where

$$\lim_{\substack{(y,t)\to(x,t_0)\\ t>t_0}} u(y,t) = \infty.$$

Throughout we assume that $u > 0$, but the assumption $u \geq 0$ would be more appropriate because of the moving boundary. We are mainly interested in m-supercaloric functions on sets where they are unbounded.

2. Preliminaries

Let Ω be an open, bounded and connected subset of $\mathbb{R}^n$ and let $0 < t_1 < t_2 < T$. We denote $\Omega_T = \Omega \times (0,T)$ and $D_{t_1,t_2} = D \times (t_1,t_2)$, where $D \subset \Omega$ is open. The parabolic boundary of a space-time cylinder D_{t_1,t_2} is $\partial_p D_{t_1,t_2} = (\overline{D} \times \{t_1\}) \cup (\partial D \times [t_1,t_2))$, that is, it consists of the initial and lateral boundaries. $D_{t_1,t_2} \Subset \Omega_T$ denotes that $\overline{D_{t_1,t_2}}$ is a compact subset of Ω_T.

We use $W^{1,p}(\Omega)$, $1 \leq p < \infty$, to denote the Sobolev space of functions $u \in L^p(\Omega)$, whose weak gradients also belong to $L^p(\Omega)$, with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

The Sobolev space with zero boundary values $W_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm of $W^{1,p}(\Omega)$. Moreover, $u \in W_{\mathrm{loc}}^{1,p}(\Omega)$ if $u \in W^{1,p}(D)$ for every $D \Subset \Omega$.

The parabolic Sobolev space $L^2(0,T;W^{1,2}(\Omega))$ consists of measurable functions $u : \Omega_T \to [-\infty,\infty]$ such that $x \mapsto u(x,t)$ belongs to $W^{1,2}(\Omega)$ for almost every $t \in (0,T)$ and

$$\iint_{\Omega_T} (|u|^2 + |\nabla u|^2)\, dx\, dt < \infty.$$

The definition for $L^2(0,T;W_0^{1,2}(\Omega))$ is similar apart from the requirement that $x \mapsto u(x,t)$ belongs to $W_0^{1,2}(\Omega)$ for almost every $t \in (0,T)$. Moreover, we say that $u \in L^2_{\mathrm{loc}}(0,T;W^{1,2}_{\mathrm{loc}}(\Omega))$ if $u \in L^2(t_1,t_2;W^{1,2}(D))$ for every $D_{t_1,t_2} \Subset \Omega_T$.

Definition 2.1. A non-negative function u is a weak supersolution to (1.1), if $u^m \in L^2_{\mathrm{loc}}(0,T;W^{1,2}_{\mathrm{loc}}(\Omega))$ and it satisfies (1.2) for every non-negative test function $\varphi \in C_0^\infty(\Omega_T)$. Similarly, u is a weak subsolution, if (1.2) holds with the inequality reversed. Moreover, u is a weak solution to (1.1), if the integral in (1.2) is zero for every test function $\varphi \in C_0^\infty(\Omega_T)$ without the sign restriction.

In order to obtain appropriate Caccioppoli type energy estimates it is convenient to impose the Sobolev space assumption on $u^{\frac{m+1}{2}}$ instead of u^m in the definition above. According to the recent result in [2], this does not make any difference for locally bounded functions.

Theorem 2.2. *Assume that u is a locally bounded non-negative function.*

(i) *If $u^{\frac{m+1}{2}} \in L^2_{loc}(0,T;W^{1,2}_{loc}(\Omega))$ satisfies* (1.2) *for every non-negative test function $\varphi \in C_0^\infty(\Omega_T)$, then $u^m \in L^2_{loc}(0,T;W^{1,2}_{loc}(\Omega)$.*

(ii) *If $u^m \in L^2_{loc}(0,T;W^{1,2}_{loc}(\Omega)$ satisfies* (1.2) *for every non-negative test function $\varphi \in C_0^\infty(\Omega_T)$, then $u^{\frac{m+1}{2}} \in L^2_{loc}(0,T;W^{1,2}_{loc}(\Omega))$.*

Remark 2.3. Under the assumption $u^{\frac{m+1}{2}} \in L^2_{\text{loc}}(0,T;W^{1,2}_{\text{loc}}(\Omega))$ the gradient in (1.2) is interpreted as

$$\nabla(u^m) = \frac{2m}{m+1} u^{\frac{m-1}{2}} \nabla(u^{\frac{m+1}{2}}).$$

We point out that it we may restrict ourselves to bounded weak supersolutions, since we consider the truncations

$$u_j = \min\{u, j\}, \quad j = 1, 2, \dots .$$

In addition, we assume that u is positive throughout, since we are interested in sets where functions are large.

We shall investigate several aspects related to unbounded m-supercaloric functions. Harnack type estimates with intrinsic geometry play a fundamental role in our study. Positive weak solutions to the porous medium equation satisfy the following intrinsic Harnack inequality, see [5, Theorem 3], [4], [8] and [15].

Lemma 2.4 (Harnack)**.** *Assume that u is a positive weak solution to* (1.1) *in Ω_T. Then there exist constants C_1 and C_2, depending on n and m, such that*

$$u(x_0, t_0) \le C_1 \inf_{x \in B(x_0, r)} u(x, t_0 + \theta),$$

where

$$\theta = \frac{C_2 \rho^2}{u(x_0, t_0)^{m-1}}$$

is such that $B(x_0, 2r) \times (t_0 - 2\theta, t_0 + 2\theta) \subset \Omega_T$.

For locally bounded positive weak supersolutions, we have the corresponding weak Harnack estimate, see [4, Theorem 17.1, p. 133] and [13].

Lemma 2.5 (Weak Harnack)**.** *Assume that u is a locally bounded positive weak supersolution to* (1.1) *in Ω_T and let $B(x_0, 8r) \times (0,T) \subset \Omega_T$. Then there exist constants C_1 and C_2, depending only on m and n, such that for almost every $t_0 \in (0,T)$, we have*

$$\fint_{B(x_0,r)} u(x,t_0)\,dx \le \left(\frac{C_1 r^2}{T-t_0}\right)^{\frac{1}{m-1}} + C_2 \operatorname*{ess\,inf}_Q u,$$

where $Q = B(x_0,4r) \times (t_0 + \frac{\theta}{2}, t_0 + \theta)$ with

$$\theta = \min\left\{T - t_0, C_1 r^2 \left(\fint_{B(x_0,r)} u(x,t_0)\,dx\right)^{-(m-1)}\right\}.$$

We shall discuss several results related to local integrability of m-supercaloric functions. Caccioppoli type energy estimates allow us to derive estimates for local integrability of the gradient in terms of the local integrability of a supersolution.

Lemma 2.6 (Caccioppoli)**.** *Assume that u is a locally bounded positive weak supersolution to* (1.1) *in Ω_T and let $\zeta \in C_0^\infty(\Omega_T)$ be a cut-off function such that $0 \le \zeta \le 1$. Then there exist numerical constants C_1 and C_2 such that*

$$\begin{aligned}&\iint_{\Omega_T} m u^{m-\varepsilon-2}\zeta^2|\nabla u|^2\,dx\,dt + \frac{1}{\varepsilon|1-\varepsilon|}\operatorname*{ess\,sup}_{t\in(0,T)}\int_\Omega u^{1-\varepsilon}\zeta^2\,dx\\ &\le \frac{C_1 m}{\varepsilon^2}\iint_{\Omega_T} u^{m-\varepsilon}|\nabla\zeta|^2\,dx\,dt + \frac{C_2}{\varepsilon|1-\varepsilon|}\iint_{\Omega_T} u^{1-\varepsilon}\zeta|\zeta_t|\,dx\,dt\end{aligned}$$

for every $\varepsilon > 0$, $\varepsilon \neq 1$.

Proof. The Caccioppoli estimate follows by choosing the test function $\varphi = u^{-\varepsilon}\zeta^2$ combined with technical smoothing and dampening arguments. For a detailed proof, we refer to [13, Lemma 2.4]. □

In the case $\varepsilon = 1$, the Caccioppoli estimate takes the following logarithmic form.

Lemma 2.7. *Assume that u is a locally bounded positive weak supersolution to* (1.1) *in Ω_T and let $\zeta \in C_0^\infty(\Omega_T)$ be a cut-off function such that $0 \le \zeta \le 1$. Then there exist numerical constants C_1 and C_2 such that*

$$\begin{aligned}&\iint_{\Omega_T} m u^{m-3}\zeta^2|\nabla u|^2\,dx\,dt + \operatorname*{ess\,sup}_{t\in(0,T)}\left|\int_\Omega \zeta^2\log u\,dx\right|\\ &\le C_1 m\iint_{\Omega_T} u^{m-1}|\nabla\zeta|^2\,dx\,dt + C_2\iint_{\Omega_T}\zeta|\zeta_t||\log u|\,dx\,dt.\end{aligned}$$

Proof. Let $0 \le t_1 < t_2 \le T$. Formally, we apply $\varphi = u^{-1}\zeta^2$ as a test function in the inequality

$$\int_{t_1}^{t_2}\int_\Omega \left(\nabla(u^m)\cdot\nabla\varphi + \varphi\frac{\partial u}{\partial t}\right)\,dx\,dt \ge 0.$$

We observe $u^{-1}\frac{\partial u}{\partial t} = \frac{\partial \log(u)}{\partial t}$ and integrate by parts to get

$$\begin{aligned}&\int_{t_1}^{t_2}\int_\Omega \Big(-mu^{m-3}|\nabla u|^2\zeta^2 + 2mu^{m-2}\zeta\nabla u\cdot\nabla\zeta - 2\zeta\zeta_t\log u\Big)\,dx\,dt\\ &+\int_\Omega \zeta(x,t_2)^2\log u(x,t_2)\,dx - \int_\Omega \zeta(x,t_1)^2\log u(x,t_1)\,dx \ge 0.\end{aligned} \tag{2.1}$$

Young's inequality gives

$$\begin{aligned}|2mu^{m-2}\zeta\nabla u\cdot\nabla\zeta| &\le 2mu^{m-2}\zeta|\nabla u||\nabla\zeta|\\ &\le \frac{m}{2}(\zeta^2u^{m-3}|\nabla u|^2 + 4u^{m-1}|\nabla\zeta|^2).\end{aligned} \tag{2.2}$$

By combining (2.1) and (2.2) we arrive at

$$\begin{aligned}&\int_{t_1}^{t_2}\int_\Omega \frac{m}{2}u^{m-3}|\nabla u|^2\zeta^2\,dx\,dt + \int_\Omega \zeta(x,t_1)^2\log u(x,t_1)\,dx\\ &\quad-\int_\Omega \zeta(x,t_2)^2\log u(x,t_2)\,dx\\ &\le 2\int_{t_1}^{t_2}\int_\Omega u^{m-1}|\nabla\zeta|^2\,dx\,dt + 2\int_{t_1}^{t_2}\int_\Omega \zeta|\zeta_t||\log u|\,dx\,dt.\end{aligned}$$

In the previous estimate we may first take supremum over t_1 and then let $t_2 \to T$ or first take supremum over t_2 and then let $t_1 \to 0$. It follows that

$$\begin{aligned}&\int_0^T\int_\Omega mu^{m-3}|\nabla u|^2\zeta^2\,dx\,dt + \operatorname*{ess\,sup}_{t\in(0,T)}\left|\int_\Omega \zeta^2\log u\,dx\right|\\ &\le 4\int_0^T\int_\Omega u^{m-1}|\nabla\zeta|^2\,dx\,dt + 4\int_0^T\int_\Omega \zeta|\zeta_t||\log u|\,dx\,dt.\end{aligned}$$

□

Finally, we recall a parabolic Sobolev's inequality, which is a tool to conclude local integrability estimates for a function in terms of its gradient.

Lemma 2.8 (Sobolev)**.** *Assume that $u \in L^p(0,T;W^{1,p}(\Omega))$ and $\zeta \in C_0^\infty(\Omega_T)$. Then there exists constant C, depending only on n, such that*

$$\iint_{\Omega_T}|\zeta u|^q\,dx\,dt \le C^q\iint_{\Omega_T}|\nabla(\zeta u)|^p\,dx\,dt\left(\operatorname*{ess\,sup}_{t\in(0,T)}\int_\Omega|\zeta u|^r\,dx\right)^{\frac{p}{n}},$$

where $r > 0$ can be chosen as we please and $q = p + \frac{pr}{n}$.

Proof. See [6, p. 7–8]. □

3. Characterizations for class $\mathfrak{B}$

In this section we consider m-supercaloric functions that have a similar behaviour as the Barenblatt solution. We say that a positive m-supercaloric function u belongs to class $\mathfrak{B}$, if $u \in L^{m-1}_{\text{loc}}(\Omega_T)$. In other words, the pressure $\frac{u^{m-1}}{m-1}$ is locally integrable. The following theorem gives several characterizations for these functions.

Theorem 3.1. *Assume that u is a positive m-supercaloric function in Ω_T. Then the following properties are equivalent:*

(i) $u \in L^q_{loc}(\Omega_T)$ *for some* $q > m-1$,
(ii) $u \in L^{m-1}_{loc}(\Omega_T)$,
(iii) $\nabla(u^m)$ *exists and* $\nabla(u^m) \in L^q_{loc}(\Omega_T)$ *whenever* $q < 1 + \frac{1}{1+mn}$,
(iv) $\displaystyle\operatorname*{ess\,sup}_{t\in(\delta,T-\delta)} \int_D u(x,t)\,dx < \infty$ *whenever* $D \Subset \Omega$ *and* $\delta \in (0, \frac{T}{2})$.

Proof. First we prove the theorem in the case, when (iv) is replaced with the following slightly weaker condition:

(iv') there exists $\alpha \in (0,1)$ such that

$$\operatorname*{ess\,sup}_{t\in(\delta,T-\delta)} \int_D u(x,t)^\alpha\,dx < \infty \tag{3.1}$$

whenever $D \Subset \Omega$ and $\delta \in (0, \frac{T}{2})$.

We show that (i) $\Longleftrightarrow$ (iii) and (i) $\Longleftrightarrow$ (iv'). The remaining equivalences are treated in Remark 5.3.

First we show that (i) implies (iii). This follows from [9, Theorem 1.4]. However, there is a missing assumption in [9, Theorem 1.4] and the existence of the Poisson modification was taken for granted in [9, Section 5]. In order to complete the proof, we show that the Poisson modification exists under the assumption $u \in L^q_{\text{loc}}(\Omega_T)$ with $q > m-1$.

Let $Q \Subset \Omega$ be a cube and let $Q' \Subset Q$ be a subcube of Q. Fix $t_1 \in (0,T)$. We redefine u by setting $u(x,t) = 0$ when $(x,t) \in Q \times (0,t_1)$. Observe that the redefined function is m-supercaloric in Q_T. By lower semicontinuity, there exists an increasing sequence of non-negative smooth functions ψ_k, such that $\psi_k \to u$ pointwise in Q_T as $k \to \infty$. Note that $\psi_k = 0$ in $Q \times (0,t_1)$ for every $k = 1, 2, \dots$.

Let h_k be the unique weak solution to the porous medium equation in $(Q \setminus \overline{Q'}) \times (0,T)$ with boundary values

$$h_k = \begin{cases} \psi_k & \text{on} \quad \partial Q' \times [0,T], \\ 0 & \text{on} \quad \partial Q \times [0,T], \\ 0 & \text{in} \quad (Q \setminus Q') \times \{0\}. \end{cases}$$

Such a solution h_k is continuous up to the boundary so that $h_k \in C((\overline{Q} \setminus Q') \times [0,T])$. By the comparison principle

$$h_1 \leq h_2 \leq \dots \quad \text{and} \quad h_k \leq u \quad \text{in} \quad (Q \setminus \overline{Q'}) \times (0,T)$$

for every $k = 1, 2, \dots$. Let $h = \lim_{k\to\infty} h_k$ and

$$v = \begin{cases} u & \text{in} \quad \overline{Q'} \times (0,T), \\ h & \text{in} \quad (Q \setminus \overline{Q'}) \times (0,T). \end{cases}$$

We claim that v is m-supercaloric in Q_T and call it the Poisson modification of u in Q_T. The crucial step in the proof is to show that $v < \infty$ on a dense subset. To this end, we show that, for any $(x_0, t_0) \in (Q \setminus \overline{Q'}) \times (0,T)$, there does not exist a sequence $(x_k, t_k) \to (x_0, t_0)$, such that

$$\lim_{k\to\infty} h_k(x_k, t_k) = \infty.$$

Suppose that such a sequence exists for some (x_0, t_0). Choose $r > 0$ so small that $B(x_0, 2r) \subset Q \setminus \overline{Q'}$ and denote

$$\theta_k = \frac{C_2 r^2}{h_k(x_k, t_k)^{m-1}}, \quad k = 1, 2, \dots,$$

where C_2 is the constant in Lemma 2.4. Since $h_k(x_k, t_k) \to \infty$ we have $\theta_k \to 0$ as $k \to \infty$. Thus, for k large enough, we have $(t_k - \theta_k, t_k + \theta_k) \subset (0,T)$. Let U be the unique positive weak solution to (1.5) with zero boundary values in $B(x_0, r)$. Such a solution is continuous up to the boundary so that $U \in C(\overline{B}(x_0, r))$. Then

$$V_k(x,t) = \frac{U(x)}{(t - t_k + (\lambda - 1)\theta_k)^{\frac{1}{m-1}}},$$

with $\lambda > 1$, is a weak solution to the porous medium equation in $B(x_0, r) \times (t_k + \theta_k, T)$. We choose $\lambda > 1$ so large that

$$\frac{\|U\|_{L^\infty(B(x_0,r))}}{(\lambda C_2 r^2)^{\frac{1}{m-1}}} \leq \frac{1}{C_1},$$

where C_1 and C_2 are the constants in Lemma 2.4. Then

$$\begin{aligned} V_k(x, t_k + \theta_k) &= \frac{U(x)}{(\lambda C_2 r^2)^{\frac{1}{m-1}}} h_k(x_k, t_k) \\ &\leq \frac{U(x)}{(\lambda C_2 r^2)^{\frac{1}{m-1}}} C_1 h_k(x, t_k + \theta_k) \\ &\leq h_k(x, t_k + \theta_k) \quad \text{for every} \quad x \in B(x_k, r), \end{aligned}$$

where we used Lemma 2.4 for the first inequality. The comparison principle implies $V_k \leq h_k$ in $B(x_k, r) \times (t_k + \theta_k, T)$. By letting $k \to \infty$ we obtain

$$\frac{U(x)}{(t - t_0)^{\frac{1}{m-1}}} \leq h(x,t) \quad \text{for every} \quad (x,t) \in B(x_0, r) \times (t_0, T)).$$

Since $h = \lim_{k\to\infty} h_k \le u$ in $(Q \setminus \overline{Q'}) \times (0,T)$, we have

$$u(x,t) \ge h(x,t) \ge \frac{U(x)}{(t-t_0)^{\frac{1}{m-1}}} \quad \text{for every} \quad (x,t) \in B(x_0,r) \times (t_0,T)$$

and thus $u \notin L^q_{\mathrm{loc}}(\Omega_T)$ whenever $q > m-1$.

Thus we may apply the Harnack type convergence theorem for weak solutions [9, Lemma 3.4] to conclude that v is m-supercaloric in Q_T as in [9, Section 5].This shows that (i) is valid.

Next we show that (iii) implies (i). Assume that ∇u^m exists and

$$\nabla u^m \in L^q_{\mathrm{loc}}(\Omega_T) \quad \text{whenever} \quad 1 \le q < 1 + \frac{1}{1+mn}.$$

In particular, $\nabla u^m \in L^1_{\mathrm{loc}}(\Omega_T)$. By the definition of a weak derivative this includes $u^m \in L^1_{\mathrm{loc}}(\Omega_T)$.

Then we show that (i) implies (iv'). Let $D \Subset \Omega$ and $\delta \in (0, \frac{T}{2})$. Again, we consider the truncations $u_j = \min\{u, j\}$, $j = 1, 2, \dots$. Since u_j is a weak supersolution, it satisfies the Caccioppoli estimate, see Lemma 2.6. By assumption, $u \in L^{m-\varepsilon}_{\mathrm{loc}}(\Omega_T)$ for some $\varepsilon \in (0,1)$. We choose a cut-off function $\zeta \in C_0^\infty(\Omega_T)$ such that $0 \le \zeta \le 1$ and $\zeta = 1$ in $D \times (\delta, T-\delta)$. Lemma 2.6 implies

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t\in(\delta,T-\delta)} \int_D u_j^{1-\varepsilon}\,dx \\
&\le C\left(\iint_{\Omega_T} u_j^{m-\varepsilon}|\nabla\zeta|^2\,dx\,dt + \iint_{\Omega_T} u_j^{1-\varepsilon}\zeta|\zeta_t|\,dx\,dt\right) \\
&\le C\left(\iint_{\Omega_T} u^{m-\varepsilon}|\nabla\zeta|^2\,dx\,dt + \iint_{\Omega_T} u^{1-\varepsilon}\zeta|\zeta_t|\,dx\,dt\right) < \infty.
\end{aligned}$$

Observe that the integrals above are finite, since $u \in L^{m-\varepsilon}_{\mathrm{loc}}(\Omega_T)$ and the support of ζ is a compact subset of Ω_T. Since the constant C is independent of j, the claim follows by letting $j \to \infty$.

Finally we show that (iv') implies (i). As above we consider the truncations of u, but this time we leave it out in the notation. Observe that all constants below are independent on the level of truncation. Let ζ be a cut-off function as above. We will show by an iteration argument that $u \in L^q_{\mathrm{loc}}(\Omega_T)$ for some $q > m-1$. The idea of the proof is the following. We will show that by (iv') we have $u \in L^{s_0}_{\mathrm{loc}}(\Omega_T)$ for $s_0 = \alpha$, and $u \in L^{s_j}_{\mathrm{loc}}(\Omega_T)$ implies $u \in L^{s_{j+1}}_{\mathrm{loc}}(\Omega_T)$ for an increasing sequence of exponents s_j. We may iterate this until either $s_j > m-1$ or $s_j = m-1$. In the former case we are done and the latter case is treated separately.

For $\alpha \in (0,1)$ we define

$$s_j = \alpha\left(1+\frac{2j}{n}\right), \quad r_j = \frac{2}{1+\frac{2j}{n}} \quad \text{and} \quad q_j = 2\left(1+\frac{2}{n\left(1+\frac{2j}{n}\right)}\right)$$

for $j = 0,1,2,\dots$. We observe that $\frac{s_j}{2}q_j = s_{j+1}$ and apply Sobolev's inequality to $w = u^{\frac{s_j}{2}}$, see Lemma 2.8. This gives

$$\begin{aligned}
&\iint_{\Omega_T} u^{s_{j+1}}\zeta^{q_j}\,dx\,dt \\
&\le C\iint_{\Omega_T}\left(u^{s_j}|\nabla\zeta|^2+\zeta^2u^{s_j-2}|\nabla u|^2\right)dx\,dt\left(\operatorname*{ess\,sup}_{t\in(0,T)}\int_\Omega \zeta^{r_j}u^\alpha\,dx\right)^{\frac{2}{n}}.
\end{aligned} \tag{3.2}$$

By (iv') we have

$$\operatorname*{ess\,sup}_{t\in(0,T)}\int_\Omega \zeta^{r_j}u^\alpha\,dx < \infty.$$

Lemma 2.6 with $\varepsilon = m - s_j$ implies

$$\begin{aligned}
&\iint_{\Omega_T}\zeta^2u^{s_j-2}|\nabla u|^2\,dx\,dt \\
&\le C\left(\iint_{\Omega_T}u^{s_j}|\nabla\zeta|^2\,dx\,dt+\iint_{\Omega_T}u^{s_j-(m-1)}\zeta|\zeta_t|\,dx\,dt\right)<\infty.
\end{aligned} \tag{3.3}$$

Observe that $u > 0$ is a lower semicontinous function and thus it attains its strictly positive minimum δ on every compact subset of Ω_T. The same δ will do for the original u and all truncations. Thus

$$u^{s_j-(m-1)} \le \delta^{-(m-1)}u^{s_j}$$

for some $\delta > 0$ in the support of ζ and the second integral on the right-hand side of (3.3) is finite. Then we consider the first integral on the right-hand side of (3.3). We note that in the first step of iteration $s_0 = \alpha$ and by (iv') we have

$$\iint_{\Omega_T}u^{s_0}|\nabla\zeta|^2\,dx\,dt \le \operatorname*{ess\,sup}_{t\in(0,T)}\int_\Omega \zeta u^\alpha\,dx < \infty.$$

which implies that $u \in L^{s_0}_{\mathrm{loc}}(\Omega_T)$. In general, from (3.2) and (3.3) we may conclude that if $u \in L^{s_j}_{\mathrm{loc}}(\Omega_T)$ for some j, then $u \in L^{s_{j+1}}_{\mathrm{loc}}(\Omega_T)$. By iterating this argument, we may step by step increase the local integrability exponent of u. It is essential that we shall use only a finite number of iterations.

This iteration can be done as long as $\varepsilon = m - s_j > 0$ and $\varepsilon = m - s_j \neq 1$. We may assume $\varepsilon > 0$ since (i) holds if $s_j > m$. The case $\varepsilon = 1$ will be treated separately. Since s_j is an increasing sequence, we can find an index k such that $s_{k-1} < m-1 \le s_k$. If $s_k > m-1$, then $u \in L^{s_k}_{\mathrm{loc}}(\Omega_T)$ and we are done. It remains to consider the case $s_k = m-1$. Denote

$$r = \frac{2\alpha}{m-1} \quad \text{and} \quad q = 2\left(1+\frac{2\alpha}{n(m-1)}\right).$$

By applying Sobolev's inequality, see Lemma 2.8, to $w = u^{\frac{m-1}{2}}$, we obtain

$$\begin{aligned}&\iint_{\Omega_T} \zeta^q u^{m-1+\frac{2\alpha}{n}}\,dx\,dt\\&\le C\iint_{\Omega_T}\Big(u^{m-1}|\nabla\zeta|^2+\zeta^2|\nabla(u^{\frac{m-1}{2}})|^2\Big)\,dx\,dt\left(\operatorname*{ess\,sup}_{t\in(0,T)}\int_\Omega \zeta^r u^\alpha\,dx\right)^{\frac{2}{n}}.\end{aligned}\tag{3.4}$$

Lemma 2.7 implies

$$\begin{aligned}&\iint_{\Omega_T}\zeta^2|\nabla(u^{\frac{m-1}{2}})|^2\,dx\,dt\\&\le C\left(\iint_{\Omega_T}u^{m-1}|\nabla\zeta|^2\,dx\,dt+\iint_{\Omega_T}|\log(u)|\zeta|\zeta_t|\,dx\,dt\right)<\infty.\end{aligned}$$

Thus the right-hand side of (3.4) is finite and $u \in L^{m-1+\frac{2\alpha}{n}}_{\mathrm{loc}}(\Omega_T)$. □

We point out some further implications related to class $\mathfrak{B}$.

Remark 3.2. A function $u \in \mathfrak{B}$ has the following properties:

(1) $u \in L^q_{\mathrm{loc}}(\Omega_T)$ for every $q < m + \frac{2}{n}$. This is a consequence of a reverse Hölder inequality for supersolutions to the porous medium equation, see [9] and [13]. In particular, this implies that $u \in L^1_{\mathrm{loc}}(\Omega_T)$.

(2) There exists a Radon measure μ on $\mathbb{R}^{n+1}$, such that u is a weak solution to the measure data problem

$$u_t - \Delta(u^m) = \mu.$$

To see this, by the discussion above $u \in L^1_{\mathrm{loc}}(\Omega_T)$ and $\nabla(u^m) \in L^1_{\mathrm{loc}}(\Omega_T)$. Thus we may apply the Riesz representation theorem to the non-negative linear operator

$$L_u(\varphi) = \iint_{\Omega_T}\left(-u\varphi_t + \nabla(u^m)\cdot\nabla\varphi\right)\,dx\,dt,$$

where $\varphi \in C_0^\infty(\Omega_T)$.

4. Characterizations for class $\mathfrak{M}$

We say that a positive m-supercaloric function u belongs to class $\mathfrak{M}$, if $u \notin L^{m-1}_{\mathrm{loc}}(\Omega_T)$. The friendly giant is a function in class $\mathfrak{M}$. The following theorem gives several characterizations for class $\mathfrak{M}$.

Theorem 4.1. *Assume that u is a positive m-supercaloric function in Ω_T. Then the following properties are equivalent:*

(i) $u \notin L^q_{loc}(\Omega_T)$ *for every* $q > m-1$,

(ii) $u \notin L^{m-1}_{loc}(\Omega_T)$,

(iii) *there exists* $\delta \in (0, \frac{T}{2})$ *such that*

$$\operatorname*{ess\,sup}_{t\in(\delta,T-\delta)} \int_D u(x,t)\,dx = \infty,$$

whenever $D \Subset \Omega$ *and* $|D| > 0$.

(iv) *there exists* $(x_0, t_0) \in \Omega_T$ *such that*

$$\liminf_{\substack{(x,t)\to(x_0,t_0)\\ t>t_0}} u(x,t)(t-t_0)^{\frac{1}{m-1}} > 0, \tag{4.1}$$

(v) *there exists* $t_0 \in (0, T)$ *such that*

$$\lim_{\substack{(x,t)\to(x_0,t_0)\\ t>t_0}} u(x,t) = \infty \quad \textit{for every} \quad x_0 \in \Omega.$$

Remark 4.2. Assume that (iii) in Theorem 4.1 does not hold and let $\alpha \in (0, 1)$. Then

$$\operatorname*{ess\,sup}_{t\in(\delta,T-\delta)} \int_D u(x,t)^\alpha\,dx \le \left(\operatorname*{ess\,sup}_{t\in(\delta,T-\delta)} \int_D u(x,t)\,dx\right)^\alpha |D|^{1-\alpha} < \infty,$$

whenever $D \times (\delta, T-\delta) \Subset \Omega_T$. This shows that (3.1) holds true and thus by Theorem 3.1 we conclude that $u \in \mathfrak{B}$.

The following lemma will be useful for us.

Lemma 4.3. *Assume that* $u > 0$ *is an* m*-supercaloric function in* Ω_T *and let* $t_0 \in (0, T)$. *Suppose that* $B(x_0, 8r) \subset \Omega$ *and that there exists a sequence* t_j *belonging to a dense subset of* (t_0, T), $j = 1, 2, \dots$, *with* $t_j \to t_0$ *as* $j \to \infty$, *such that*

$$\lim_{j\to\infty} \int_{B(x_0,r)} u(x, t_j)\,dx = \infty.$$

Then there exists C, *depending only on* n *and* m, *such that*

$$u(x,t) \ge C \left(\frac{r^2}{t-t_0}\right)^{\frac{1}{m-1}} \quad \textit{for every} \quad (x,t) \in B(x_0, 4r) \times (t_0, T).$$

Proof. Denote

$$u_\lambda(x,t) = \min\{u(x,t), \lambda\} \quad \text{with} \quad \lambda > 0.$$

By [9, Theorem 3.2] u_λ is a weak supersolution in Ω_T for every $\lambda > 0$. Let $s > t_0$ to be chosen so that $s - t_0$ is small enough. We assume that the times $t_j \in (t_0, T)$, $j = 1, 2, \dots$ belong to the dense subset of $(0, T)$ where Lemma 2.5 is applicable. Furthermore, we may assume that

$$\fint_{B(x_0,r)} u(x, t_j)\,dx > 2 \left(\frac{C_1 r^2}{s - t_0}\right)^{\frac{1}{m-1}}.$$

Here C_1 is the constant in Lemma 2.5. Choose λ_j such that

$$\fint_{B(x_0,r)} u_{\lambda_j}(x,t_j)\,dx = 2\left(\frac{C_1 r^2}{s-t_0}\right)^{\frac{1}{m-1}}.$$

Apply Lemma 2.5 to u_{λ_j} at time t_j to obtain

$$2\left(\frac{C_1 r^2}{s-t_0}\right)^{\frac{1}{m-1}} \le \left(\frac{C_1 r^2}{s-t_j}\right)^{\frac{1}{m-1}} + C_2 \inf_{Q_j} u_{\lambda_j},$$

where

$$Q_j = B(x_0,4r)\times\left(t_j+\frac{\theta_j}{2}, t_j+\theta_j\right) \quad \text{and} \quad \theta_j = \min\left\{s-t_j, \frac{s-t_0}{2^{m-1}}\right\},$$

$j = 1,2,\dots$. By letting $j\to\infty$, we have

$$u(x,t) \ge \frac{1}{C_2}\left(\frac{C_1 r^2}{s-t_0}\right)^{\frac{1}{m-1}} \ge \frac{1}{C_2}\left(\frac{C_1 r^2}{2^m(t-t_0)}\right)^{\frac{1}{m-1}} \tag{4.2}$$

for every $(x,t)\in B(x_0,4r)\times\big(t_0+\frac{s-t_0}{2^m}, t_0+\frac{s-t_0}{2^{m-1}}\big)$. Finally we observe that for every $t\in(t_0,T)$ we may choose $s>t_0$ such that $t\in\big(t_0+\frac{s-t_0}{2^m}, t_0+\frac{s-t_0}{2^{m-1}}\big)$ and thus (4.2) holds for every $(x,t)\in B(x_0,4r)\times(t_0,T)$. □

Proof of Theorem 4.1. We show that (i) $\iff$ (iii) and (iii) $\implies$ (iv) $\implies$ (v) $\implies$ (iii). For the remaining equivalences, see Remark 5.3.

The claim that (i) and (iii) are equivalent follows from Theorem 3.1 and Lemma 4.3.

We show that (iii) implies (iv). Assume that

$$\operatorname*{ess\,sup}_{t\in(\delta,T-\delta)} \int_{B(x_0,r)} u(x,t)\,dx = \infty.$$

Then we may choose a sequence t_j, $j=1,2,\dots$ belonging to the dense subset of $(0,T)$ where Lemma 4.3 is applicable, with $t_j\to t_0$ as $j\to\infty$, such that

$$\lim_{j\to\infty}\int_{B(x_0,r)} u(x,t_j)\,dx = \infty.$$

By Lemma 4.3

$$u(x,t)\ge C\left(\frac{r^2}{t-t_0}\right)^{\frac{1}{m-1}} \quad \text{for every} \quad (x,t)\in B(x_0,4r)\times(t_0,T).$$

This implies (4.1).

Then we show that (iv) implies (v). Assume that there exists $(x_0,t_0)\in\Omega_T$ such that (4.1) holds. Then there exist $r>0$, $\delta>0$ and $\varepsilon>0$ such that

$$(t-t_0)^{\frac{1}{m-1}} u(x,t)\ge\varepsilon \quad \text{for every} \quad (x,t)\in B(x_0,r)\times(t_0,t_0+\delta).$$

In particular

$$\fint_{B(x_0,r)} u(x,t)\,dx \geq \varepsilon(t-t_0)^{-\frac{1}{m-1}} \quad \text{for every} \quad t \in (t_0, t_0+\delta).$$

Lemma 4.3 shows that

$$u(x,t) \geq C\left(\frac{r^2}{t-t_0}\right)^{\frac{1}{m-1}} \quad \text{for every} \quad (x,t) \in B(x_0,4r) \times (t_0, t_0+\delta).$$

Thus $B(x_0,4r) \subset \Xi^{\perp}(t_0)$, where

$$\Xi^{\perp}(t_0) = \Big\{x_0 \in \Omega : \lim_{\substack{(x,t)\to(x_0,t_0)\\ t>t_0}} u(x,t) = \infty\Big\}.$$

We may repeat the same argument for any ball intersecting $B(x_0,4r)$. Therefore, choosing a suitable chain of balls, we can reach any point in Ω and conclude that $\Xi^{\perp}(t_0) = \Omega$.

Finally we show that (v) implies (iii). If $\Xi^{\perp}(t_0) = \Omega$ for some $t_0 \in (0,T)$, we have

$$\int_D u(x,t)\,dx \to \infty \quad \text{as} \quad t \to t_0+,$$

for every set $D \Subset \Omega$ with $|D| > 0$. Hence there is $\delta > 0$ such that

$$\operatorname*{ess\,sup}_{t\in(\delta,T-\delta)} \int_D u(x,t)\,dx = \infty.$$

□

Remark 4.4. The friendly giant plays an important role as a minorant for m-supercaloric functions which blow up at time t_0. Assume that u is a non-negative m-supercaloric function in Ω_T with the property that

$$\lim_{\substack{(y,t)\to(x,0)\\ t>0}} u(y,t) = \infty \quad \text{for every} \quad x \in \Omega.$$

Let $\sigma > 0$. The comparison principle gives

$$u(x,t) \geq U(x)(t+\sigma)^{-\frac{1}{m-1}} \quad \text{for every} \quad (x,t) \in \Omega_{T-\sigma},$$

where U is a solution to (1.5) as in the construction of the friendly giant. By letting $\sigma \to 0$, we have

$$u(x,t) \geq U(x)t^{-\frac{1}{m-1}} \quad \text{for every} \quad (x,t) \in \Omega_T.$$

In particular,

$$\liminf_{\substack{(y,t)\to(x,0)\\ t>0}} u(y,t)t^{\frac{1}{m-1}} > 0 \quad \text{for every} \quad x \in \Omega.$$

This shows that an m-supercaloric function, with infinite initial values on the whole time slice $\Omega \times \{0\}$, blows up at a rate greater or equal to $t^{\frac{1}{m-1}}$, see [14, p. 111–114].

The next example shows that an m-supercaloric function may blow up faster than the friendly giant.

Example 4.5. Let

$$V(x,t) = U(x) e^{\frac{1}{(m-1)t}}, \quad t > 0.$$

Here U is a solution to (1.5) as in the construction of the friendly giant. We will show that V is a supersolution. A straightforward computation gives

$$V_t(x,t) - \Delta(V(x,t)^m) = e^{\frac{1}{(m-1)t}} \left(e^{\frac{1}{t}} - \frac{1}{t^2} \right) \frac{U(x)}{m-1} \geq 0.$$

In a similar manner, we can construct supersolutions that blow up even faster. Let $f : (0,\infty) \to (0,\infty)$ and define

$$V(x,t) = U(x) e^{\frac{f(t)}{m-1}}, \quad t > 0.$$

Then V is a supersolution, if f satisfies $f'(t) + e^{f(t)} \geq 0$. By choosing f in an appropriate way, we see that for any $\varepsilon > 0$, we have an m-supercaloric function V for which $V \notin L^{\varepsilon}_{\text{loc}}(\Omega_T)$ and $\nabla V \notin L^{\varepsilon}_{\text{loc}}(\Omega_T)$.

Next we give an explicit example of the dichotomy between classes $\mathfrak{B}$ and $\mathfrak{M}$ by constructing an m-supercaloric function as a limit of a sequence of solutions to initial value problems. Depending on the choice of the initial values, the solutions either converge to a Barenblatt type solution, or the limit solution blows up at a rate of the friendly giant.

Example 4.6. For $k = 1, 2, \dots$, consider a weak solution with zero lateral boundary values to the problem

$$\begin{cases} \partial_t u_k - \Delta(u_k^m) = 0 \quad \text{in} \quad B(0,1) \times (0,\infty), \\ u_k(x,0) = a_k \chi_{B(0,\frac{1}{k})} \quad \text{for every} \quad x \in B(0,1). \end{cases}$$

Set

$$v(x,t) = \frac{u_k(x, a_k^{1-m} t)}{a_k},$$

where a_k are to be chosen later. The function v satisfies

$$\begin{cases} \partial_t v - \Delta(v^m) = 0 \quad \text{in} \quad B(0,1) \times (0,\infty), \\ v(x,0) = \chi_{B(0,\frac{1}{k})}(x) \quad \text{for every} \quad x \in B(0,1). \end{cases}$$

Our aim is to compare v to the Barenblatt solution in a suitable space-time cylinder. Let

$$\mathcal{B}(x,t) = (t+t_0)^{-\lambda} \left(C - \frac{\lambda(m-1)}{2mn} \frac{|x|^2}{(t+t_0)^{\frac{2\lambda}{n}}} \right)_+^{\frac{1}{m-1}},$$

where $\lambda = \frac{n}{n(m-1)+2}$. We choose

$$C = C_0 \frac{\lambda(m-1)}{2mn} \frac{1}{k^{\frac{2\beta\lambda}{n}}} \quad \text{and} \quad t_0 = C_0^{-\frac{n}{2\lambda}} k^{\beta - \frac{n}{\lambda}}.$$

Here $\beta = (m-1)n$ and C_0 is chosen in such a way that $\mathcal{B}(0,0) \leq 1$.

For $x \in \partial B\left(0, \frac{1}{k}\right)$, we have

$$\mathcal{B}(x,0) = t_0^{-\lambda}\left(C - \frac{\lambda(m-1)}{2mn}\frac{k^{-2}}{t_0^{\frac{2\lambda}{n}}}\right)_+^{\frac{1}{m-1}} = 0,$$

since

$$C - \frac{\lambda(m-1)}{2mn}\frac{k^{-2}}{t_0^{\frac{2\lambda}{n}}} = C_0\frac{\lambda(m-1)}{2mn}\left(\frac{1}{k^{\frac{2\beta\lambda}{n}}} - \frac{k^{-2}}{k^{(\beta-\frac{n}{\lambda})\frac{2\lambda}{n}}}\right) = 0.$$

Then $\mathcal{B}(\cdot,0) \leq 1$ in $B\left(0,\frac{1}{k}\right)$ and $\mathcal{B}(\cdot,0) = 0$ in $B(0,1) \setminus B\left(0,\frac{1}{k}\right)$, which implies $\mathcal{B} \leq v$ in $B(0,1) \times \{0\}$. Next, we want to find maximal $\theta > 0$ such that $\mathcal{B}$ takes zero lateral boundary values in $B(0,1) \times (0,\theta)$. By solving θ from

$$C - \frac{\lambda(m-1)}{2mn}\frac{1}{(\theta+t_0)^{\frac{2\lambda}{n}}} = 0$$

we have

$$\theta = C_0^{-\frac{n}{2\lambda}}k^{\beta}(1-k^{\frac{n}{\lambda}}).$$

Then, by the comparison principle, $v \geq \mathcal{B}$ in $B(0,1) \times (0,\theta)$. We observe that

$$\mathcal{B}(x,\theta) \geq ck^{-\lambda\beta}C^{\frac{1}{m-1}} = ck^{-\lambda\beta\left(1+\frac{2}{n(m-1)}\right)} \quad \text{for } x \in B\left(0,\tfrac{1}{4}\right).$$

Here $c = c(C_0,m,n)$. By switching back to the original variables we arrive at

$$u_k(x,\theta a_k^{1-m}) \geq ca_k k^{-\lambda\beta\left(1+\frac{2}{n(m-1)}\right)} = ca_k k^{\frac{-\beta}{m-1}}.$$

Choosing $T = \theta a_k^{1-m}$ gives

$$u_k(x,T) \geq cT^{-\frac{1}{m-1}}.$$

We consider two cases. If

$$\frac{k^{\beta}}{a_k^{m-1}} = \left(\frac{k^n}{a_k}\right)^{m-1} \to 0$$

as $k \to \infty$, we have

$$u(x,T) \geq cT^{-\frac{1}{m-1}} \quad \text{for every } T > 0,$$

and therefore u is in class $\mathfrak{M}$.

On the other hand, if

$$\frac{a_k}{k^n} \to a < \infty,$$

as $k \to \infty$, then

$$\int_{B(0,1)} a_k\chi_{B(0,\frac{1}{k})}(x)\varphi(x)\,dx \to a\varphi(0)$$

for all $\varphi \in C_0^\infty(B(0,1))$, showing that u attains the initial value $a\delta$. Thus u is a Barenblatt type solution, which implies that u is in class $\mathfrak{B}$.

5. Infinity sets

Assume that $u > 0$ is an m-supercaloric function in Ω_T. We consider two sets at time $t_0 \in (0,T)$. We recall the infinity set that we already encountered in the proof of Theorem 4.1 defined as

$$\Xi^{\perp}(t_0) = \Big\{ x_0 \in \Omega : \lim_{\substack{(x,t)\to(x_0,t_0)\\ t>t_0}} u(x,t) = \infty \Big\}.$$

In addition, we consider yet another infinity set

$$\Xi^{\downarrow}(t_0) = \Big\{ x_0 \in \Omega : \lim_{\substack{t\to t_0\\ t>t_0}} u(x_0,t) = \infty \Big\}.$$

The difference is that in the latter set the limit is taken vertically. For both sets the times $t \le t_0$ are excluded in the limit procedure. Clearly $\Xi^{\perp}(t_0) \subset \Xi^{\downarrow}(t_0)$, but the sets are not necessarily same. This can be seen by considering the Barenblatt solution. In this case $\Xi^{\perp}(0) = \emptyset$, but $\Xi^{\downarrow}(0) = \{0\}$. There is an interesting phenomenon: even though the sets may be different, either they both are of n-dimensional measure zero, or they occupy the whole time slice. Moreover, the latter alternative cannot occur for $\Omega = \mathbb{R}^n$.

Theorem 5.1. *Assume that u is a positive m-supercaloric function in Ω_T. Then for every $t \in (0,T)$ there are two alternatives:*

$$\textit{either} \quad |\Xi^{\downarrow}(t)| = |\Xi^{\perp}(t)| = 0 \quad \textit{or} \quad \Xi^{\downarrow}(t) = \Xi^{\perp}(t) = \Omega.$$

Proof. Let $t_0 \in (0,T)$. Since $\Xi^{\perp}(t_0) \subset \Xi^{\downarrow}(t_0)$, it suffices to show, that if $|\Xi^{\downarrow}(t_0)| > 0$, then $\Xi^{\perp}(t_0) = \Omega$. Suppose that $|\Xi^{\downarrow}(t_0)| > 0$. Then there exist $x_0 \in \Omega$ and $r > 0$ such that $B(x_0, 8r) \subset \Omega$ and $|\Xi^{\downarrow}(t_0) \cap B(x_0,r)| > 0$. Let $k = 1, 2, \ldots$ and $x \in \Xi^{\downarrow}(t_0)$. By definition of the set $\Xi^{\downarrow}(t_0)$, there exists $t_x^k \in (t_0, T)$ such that $u(x,t) > k$ for every $t \in (t_0, t_x^k)$. Let

$$E_k = \bigcup \Big\{ \{x\} \times (0, t_x^k) : x \in \Xi^{\downarrow}(t_0) \Big\}$$

and

$$E_k(t) = \{x \in B(x_0,r) : (x,t) \in E_k\}, \quad t \in (t_0,T).$$

Observe, that $E_k(t)$ is the projection of E_k to $B(x_0,r)$ and $E_k(t) \subset \Xi^{\downarrow}(t_0)$. It is clear that

$$\Xi^{\downarrow}(t_0) \cap B(x_0,r) = \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{\infty} E_k\left(\frac{1}{j}\right).$$

For a fixed k, $E_k(\frac{1}{j})$, $j = 1, 2, \dots$, is a monotonically increasing sequence of sets. Thus

$$|\Xi^{\downarrow}(t_0) \cap B(x_0, r)| \le \left| \bigcup_{j=1}^{\infty} E_k\left(\frac{1}{j}\right) \right| = \lim_{j\to\infty} \left| E_k\left(\frac{1}{j}\right) \right|.$$

Consequently, there exists an index j_k such that

$$|E_k(t)| \ge \left| E_k\left(\frac{1}{j_k}\right) \right| \ge \frac{1}{2}|\Xi^{\downarrow}(t_0) \cap B(x_0, r)| > 0 \quad \text{for every} \quad t \in \left(t_0, \frac{1}{j_k}\right).$$

We may choose a time $t_j \in \left(t_0, \frac{1}{j_k}\right)$ for every $j = 1, 2, \dots$ as in Lemma 4.3 and conclude

$$\int_{B(x_0,r)} u(x, t_j)\, dx \ge \int_{E_j(t_j)} u(x, t_j)\, dx \ge \frac{j}{2}|\Xi^{\downarrow}(t_0) \cap B(x_0, r)| \to \infty,$$

as $j \to \infty$. By Lemma 4.3 we have $B(x_0, 2r) \subset \Xi^{\perp}(t_0)$. Thus the infinity set has expanded. Finally a chaining shows that $\Xi^{\perp}(t_0) = \Omega$. □

As a consequence, we obtain characterizations for classes $\mathfrak{B}$ and $\mathfrak{M}$ in terms of the infinity sets.

Corollary 5.2. *Assume that u is a positive m-supercaloric function in Ω_T. Then*

(i) *$u \in \mathfrak{B}$ if and only if $|\Xi^{\downarrow}(t)| = 0$ for every $t \in (0, T)$ and*
(ii) *$u \in \mathfrak{M}$ if and only if $|\Xi^{\downarrow}(t)| > 0$ for some $t \in (0, T)$.*

The corresponding claims also hold true for $\Xi^{\perp}(t)$.

Proof. Claim (ii) is a restatement of Theorem 4.1 (v) by taking into account Theorem 5.1. Claim (i) follows immediately since classes $\mathfrak{B}$ and $\mathfrak{M}$ are mutually exclusive. □

Remark 5.3. We show the equivalence of (i) and (ii) in Theorem 3.1. It is clear that (i) implies (ii). We show the opposite implication by contradiction. Suppose that $u \notin \mathcal{B}$. Then $u \in \mathfrak{M}$. By claim (iv) in Theorem 4.1, there exists $(x_0, t_0) \in \Omega_T$, such that

$$u(x, t) \ge C(t - t_0)^{-\frac{1}{m-1}} \quad \text{in a neighbourhood of } (x_0, t_0).$$

This implies $u \notin L^{m-1}_{loc}(\Omega_T)$. By a similar reasoning, we can include the endpoint $\alpha = 1$ in (3.1), concluding the equivalence of (iv) and (iv') in Theorem 3.1. By Hölder's inequality (iv) implies (iv'). Again, we shall show the opposite implication by contradiction. If (iv) does not hold, Theorem 4.1 implies that $u \in \mathfrak{M}$ and therefore $u \notin \mathfrak{B}$. Thus (iv') does not hold.

Finally, we show that m-supercaloric functions with infinity sets of non-zero measure exist only in the case when the domain Ω is bounded. Here the global bound $u > 0$ is decisive.

Theorem 5.4. *If $u : \mathbb{R}^n \times (0,T) \to (0,\infty]$ is m-supercaloric, then $u \in \mathfrak{B}$.*

Proof. We make a counter assumption: there exists a time $t_0 \in (0,T)$, such that $|\Xi^\perp(t_0)| > 0$. Then, by Theorem 5.1, we have $\Xi^\perp(t_0) = \mathbb{R}^n$. We consider the friendly giant (1.4) with $\Omega = B(0,1)$ and see that

$$V(x,t) = U(x)(t-t_0)^{-\frac{1}{m-1}}$$

is a weak solution to the porous medium equation in $B(0,1) \times (t_0,T)$. Let $r > 0$. By the scaling property of solutions, the function

$$U\left(\frac{x}{r}\right)\left(\frac{r^2}{t-t_0}\right)^{\frac{1}{m-1}}$$

is a weak solution in $B(0,r) \times (t_0,T)$. By the comparison principle

$$u(x,t) \geq U\left(\frac{x}{r}\right)\left(\frac{r^2}{t-t_0}\right)^{\frac{1}{m-1}} \quad \text{for every} \quad (x,t) \in B(0,r) \times (t_0,T).$$

Note that

$$\eta = \inf_{x \in B(0,\frac{1}{2})} U(x) > 0.$$

Thus

$$u(x,t) \geq \eta \left(\frac{r^2}{t-t_0}\right)^{\frac{1}{m-1}} \quad \text{for every} \quad (x,t) \in B\left(0,\frac{r}{2}\right) \times (t_0,T).$$

By letting $r \to \infty$, we conclude $u \equiv \infty$. This is a contradiction, as u was assumed to be finite in a dense subset. Hence $|\Xi^\perp(t)| = 0$ for every $t \in (0,T)$ and thus $u \in \mathfrak{B}$. □

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(J. Kinnunen) Department of Mathematics, Aalto University, P. O. Box 11100, FI-00076 Aalto University, Finland

E-mail address: `juha.k.kinnunen@aalto.fi`

(P. Lehtelä) Department of Mathematics, Aalto University, P. O. Box 11100, FI-00076 Aalto University, Finland

E-mail address: `pekka.lehtela@aalto.fi`

(P. Lindqvist) Department of Mathematics, Norwegian University of Science and Technology, N-7491 Trondheim, Norway

E-mail address: `peter.lindqvist@ntnu.no`

(M. Parviainen) University of Jyvaskylä, Department of Mathematics and Statistics, P. O. Box 35, FI-40014 University of Jyvaskylä, Finland

E-mail address: `mikko.j.parvianen@jyu.fi`