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Bayesian Gaussian mixture linear inversion in geophysical inverse problems

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SUMMARY

We present a Bayesian linear inversion based on Gaussian mixture models and its application to geophysical inverse problems. The proposed inverse method is based on a Bayesian approach where we assume a Gaussian mixture random field for the prior model and a Gaussian linear likelihood function. The model for the latent discrete variable is defined to be a stationary first-order Markov chain. Here, we propose an analytical solution of the posterior distribution of the inverse problem. A sampling algorithm can be used to simulate realizations from the posterior model. Two examples of applications using real data are presented. The first example is a rock physics inversion for the estimation of facies and porosity; the second example is a seismic inversion for the estimation of facies and P-impedance. For each example, we show a set of conditional simulations, and the corresponding maximum a posteriori and prediction intervals.

Introduction

Bayesian inversion methods are commonly used in geophysics and petroleum engineering for solving inverse problems and estimating an unknown model from measured data in the subsurface (Tarantola, 2005). Two common assumptions in Bayesian inversion are the Gaussian prior distribution of the model and the linearity of the physical relation that links the model to the data. These two assumptions are not necessary but allows one to analytically compute the solution of the Bayesian inverse problem. Otherwise, Markov chain Monte Carlo (McMC) methods can be used to sample from the prior and accept or reject the proposed model according to the likelihood of observing the measured data from the proposed model. Examples in seismic and electromagnetic data inversion can be found in Buland and Omre (2003) and Buland and Kolbjørnsen (2012). Most of the physical models used in geophysics, such as seismic convolution or rock physics relations, are linear or can be linearized (see, for example, Aki and Richards, 1980, and Mavko et al., 2009). However, most of the properties in the subsurface, such as elastic attributes, porosity, or permeability, are not generally Gaussian, but show a multimodal behaviour due to the different rock and fluid properties of the different rock types or facies (Grana and Della Rossa, 2010). For example, porosity in a mixture of sand and shale is generally bimodal. In this work, we propose a Bayesian inversion method under the assumptions that the prior distribution is a Gaussian mixture model (i.e., a linear combination of Gaussian distributions) and the likelihood model is linear with additive Gaussian errors (Gaussian linear likelihood). We show two examples: the first example is a rock physics model where the operator is a linear relation that links porosity to velocity; the second example is a seismic model where the operator is a convolutional model that links velocity to seismic amplitudes.

Method and Theory

The estimation of rock and fluid properties from geophysical data in the subsurface is an inverse problem. Indeed, if $\boldsymbol{m}$ represents the model (n_m x 1 vector) to be estimated and $\boldsymbol{d}$ the measured data (n_d x 1 vector), the inverse problem can be formulated as:

$$\boldsymbol{d} = \mathbf{G}\boldsymbol{m} + \boldsymbol{\varepsilon} \tag{1}$$

where $\mathbf{G}$ is the n_d x n_m matrix associated to the linear operator from the model space to the data space and $\boldsymbol{\varepsilon}$ is the zero-mean Gaussian error (n_d x 1 vector). Hence, the likelihood function is Gaussian linear, with $p(\boldsymbol{d}|\boldsymbol{m}) = \varphi_{n_d}(\boldsymbol{d};\mathbf{G}\boldsymbol{m},\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}})$, where φ_n is the n-dimensional Gaussian probability density function. In a probabilistic setting, finding the solution of the inverse problem in Eq. 1 corresponds to assessing the posterior probability distribution of $\boldsymbol{m}|\boldsymbol{d}$. According to Bayes' rule, the posterior distribution $p(\boldsymbol{m}|\boldsymbol{d})$ can be expressed as:

$$p(\boldsymbol{m}|\boldsymbol{d}) = \frac{p(\boldsymbol{d}|\boldsymbol{m})p(\boldsymbol{m})}{p(\boldsymbol{d})}, \tag{2}$$

where $p(\boldsymbol{d}|\boldsymbol{m})$ is the likelihood function, $p(\boldsymbol{m})$ is the prior distribution of the model, and $p(\boldsymbol{d})$ is a normalizing constant. The shape of the posterior distribution depends on the prior and the likelihood function. If we assume a Gaussian linear likelihood and a Gaussian prior model, the posterior $p(\boldsymbol{m}|\boldsymbol{d})$ is a Gaussian distribution and the expressions of the conditional mean and the conditional covariance matrix can be analytically computed. This approach can be extended to other classes, for example Gaussian mixture random fields (Grana and Della Rossa, 2010). A Gaussian mixture model can be written as:

$$p(\boldsymbol{m}) = \sum_{\boldsymbol{\kappa}\in\Omega_{\kappa}^{n_m}} p(\boldsymbol{m}|\boldsymbol{\kappa})p(\boldsymbol{\kappa}) = \sum_{\boldsymbol{\kappa}\in\Omega_{\kappa}^{n_m}} \varphi_{n_m}(\boldsymbol{m};\boldsymbol{\mu}_{\boldsymbol{m}|\boldsymbol{\kappa}},\boldsymbol{\Sigma}_{\boldsymbol{m}|\boldsymbol{\kappa}})p(\boldsymbol{\kappa}), \tag{3}$$

where $\boldsymbol{\kappa}$ represents the latent discrete variable along the n_m x 1 vector with $\boldsymbol{\kappa} \in \Omega_\kappa = \{1,...,K\}$. In geophysical applications, this variable can represent rock types or facies. We assume that $\boldsymbol{\Sigma}_{m|\kappa} = \boldsymbol{\Sigma}_{m|\kappa}^{\sigma^{1/2}} \boldsymbol{\Sigma}_{m|\kappa}^{\rho} \boldsymbol{\Sigma}_{m|\kappa}^{\sigma^{1/2}}$ where $\boldsymbol{\Sigma}_{m|\kappa}^{\sigma^{1/2}}$ is the diagonal standard deviation matrix and $\boldsymbol{\Sigma}_{m|\kappa}^{\rho}$ is the spatial correlation matrix obtained from a spatial correlation function $\rho_{\boldsymbol{m}}(\Delta)$, that represents a model with full correlation between the residuals in all Gaussian components of the mixture.
The model for the discrete latent variable may be defined to be a stationary first-order Markov chain:

$$p(\boldsymbol{\kappa}) = p(\kappa_1)\prod_{t=2}^{n_m} p(\kappa_t \mid \kappa_{t\text{-}1}) \tag{4}$$

with shift invariant transition matrix $\mathbf{P}_\kappa$, of size K x K, containing the transition probabilities $p(\kappa_t \mid \kappa_{t\text{-}1})$ and $p(\kappa_1) = p_s(\kappa_1)$ where p_s represents the stationary distribution of $\mathbf{P}_{\boldsymbol{\kappa}}$.
For a Gaussian linear likelihood function $p(\boldsymbol{d} \mid \boldsymbol{m})$ as defined in Eq. 1, it is known that the posterior $p(\boldsymbol{m} \mid \boldsymbol{d})$ is also a Gaussian mixture. Indeed, if we assume that $\boldsymbol{m}$ is distributed according to a Gaussian mixture model as in Eq. 3 and $\boldsymbol{d}$ is obtained as a linear transformation $\mathbf{G}$ of $\boldsymbol{m}$ as in Eq. 1 with $p(\boldsymbol{\varepsilon}) = \varphi_{n_d}(\mathbf{0}, \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}})$ and Gaussian linear likelihood function $p(\boldsymbol{d}|\boldsymbol{m}) = \varphi_{n_d}(\boldsymbol{d}; \mathbf{G}\boldsymbol{m}, \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}})$, then the posterior is a Gaussian mixture model

$$p(\boldsymbol{m} \mid \boldsymbol{d}) = \sum_{\boldsymbol{\kappa} \in \Omega_k^{n_m}} \varphi_{n_m}(\boldsymbol{m}; \boldsymbol{\mu}_{m|d,\kappa}, \boldsymbol{\Sigma}_{m|d,\kappa}) p(\boldsymbol{\kappa} \mid \boldsymbol{d}), \tag{5}$$

where

$$\boldsymbol{\mu}_{m|d,\kappa} = \boldsymbol{\mu}_{m|\kappa} + \boldsymbol{\Sigma}_{m|\kappa}\mathbf{G}^T\left(\mathbf{G}\boldsymbol{\Sigma}_{m|\kappa}\mathbf{G}^T + \boldsymbol{\Sigma}_\varepsilon\right)^{-1}\left(\boldsymbol{d} - \mathbf{G}\boldsymbol{\mu}_{m|\kappa}\right) \tag{6}$$

$$\boldsymbol{\Sigma}_{m|d,\kappa} = \boldsymbol{\Sigma}_{m|\kappa} - \boldsymbol{\Sigma}_{m|\kappa}\mathbf{G}^T\left(\mathbf{G}\boldsymbol{\Sigma}_{m|\kappa}\mathbf{G}^T + \boldsymbol{\Sigma}_\varepsilon\right)^{-1}\mathbf{G}\boldsymbol{\Sigma}_{m|\kappa}. \tag{7}$$

The posterior $p(\boldsymbol{\kappa} \mid \boldsymbol{d})$ cannot be exactly assessed due to a computer demanding normalizing constant. A reliable approximation $p^*(\boldsymbol{\kappa} \mid \boldsymbol{d})$ can however be obtained (Rimstad and Omre, 2013). Using this approximation as proposal in an indepenent proposal McMC algorithm provides realizations from $p(\boldsymbol{\kappa} \mid \boldsymbol{d})$ with reasonable acceptance rates. Hence, realizations from the posterior of interest, $p(\boldsymbol{m} \mid \boldsymbol{d})$, can be obtained. From these realizations, maximum posterior predictions and prediction intervals can be assessed.

Examples

We present two applications using a well log dataset including a facies classification (sand, shale, and silt), a well log of P-impedance (I_{P}), a porosity curve (ϕ) computed from density, and the collocated synthetic seismic trace at the well location. The data are shown in Figure 1. The histograms of porosity and P-impedance are multimodal and can be described by 3-component Gaussian mixture models. In the first example, the likelihood function is based on a linear regression, $I_P = -14\phi + 10$, with error term with standard deviation 10^{-6}. In this example, we aim to estimate porosity and facies from the measured P-wave velocity. In the second example, we use a convolutional likelihood model and we aim to estimate P-impedance and facies from the collocated synthetic seismic trace at the well location. The Gaussian mixture prior distributions used for the two inversions are shown in Figure 1 (right plots). The transition matrix $\mathbf{P}_\kappa$ was estimated from well logs.

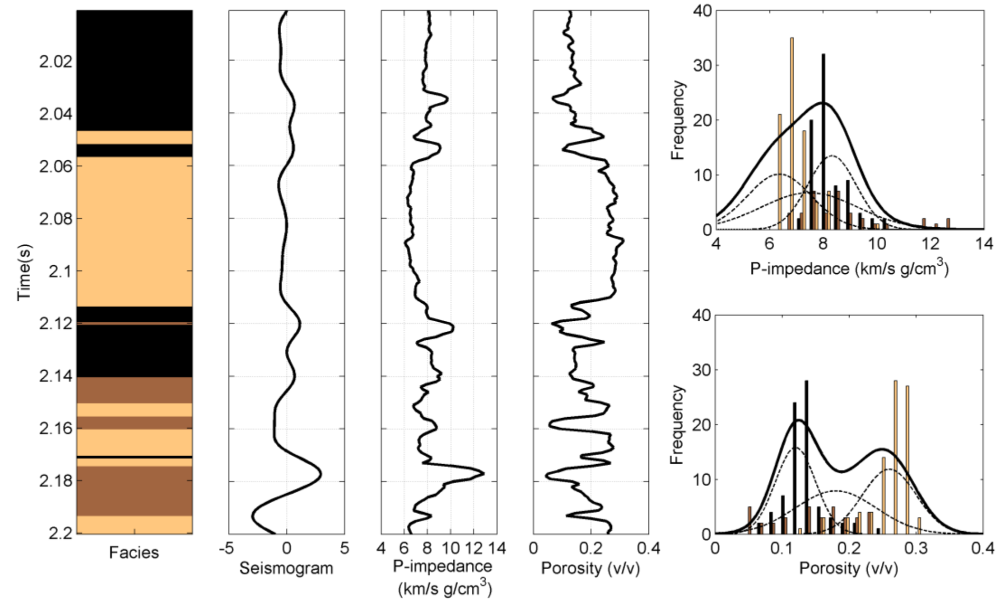

Figure 1 *Well log dataset, from left to right: log-facies classification (sand in yellow, silt in brown, shale in black); seismogram; P-impedance log; porosity curve; and histograms of P-impedance and porosity. On the right plots, we also show the prior distributions (solid lines) used in the Bayesian inversion (the dashed lines represent the weighted Gaussian components).*

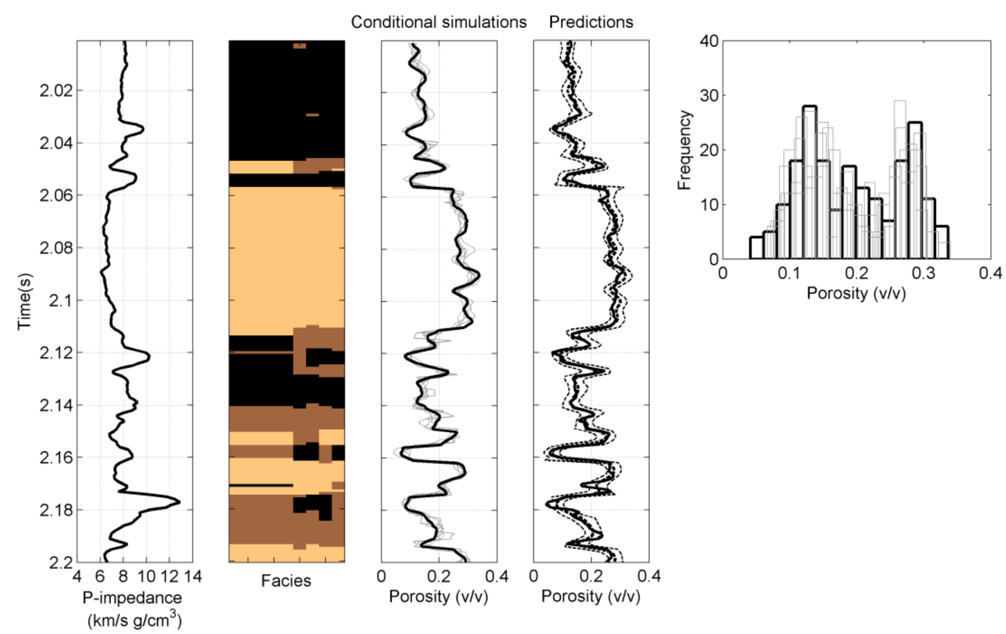

Figure 2 *Bayesian Gaussian mixture inversion of P-impedance data for porosity estimation, from left to right: actual P-impedance log (data* ***d*** *in Example 1); facies classification (the first column represents the actual log-facies profile, the other columns show 4 conditional simulations); porosity simulations (the black line shows to the first conditional simulation corresponding to the first facies simulation in the facies plot, grey lines represent four other simulations); predictions (the solid line represents the actual porosity curve, the dashed lines represent the maximum a posteriori (MAP) and the 90% prediction interval); and histograms of conditional simulations.*

In Figure 2, we show the results of the Bayesian Gaussian mixture inversion of the actual P-impedance log to estimate porosity and the underlying facies. Similarly to the actual data, the histograms of the conditional simulations are multimodal. In Figure 3, we show the results of the Bayesian Gaussian mixture inversion of the collocated synthetic seismic trace to estimate P-impedance and the latent categorical facies. In this case, the Gaussian components of the mixture are generally skewed and a log-Normal transformation is required, as visible in the true model as well.

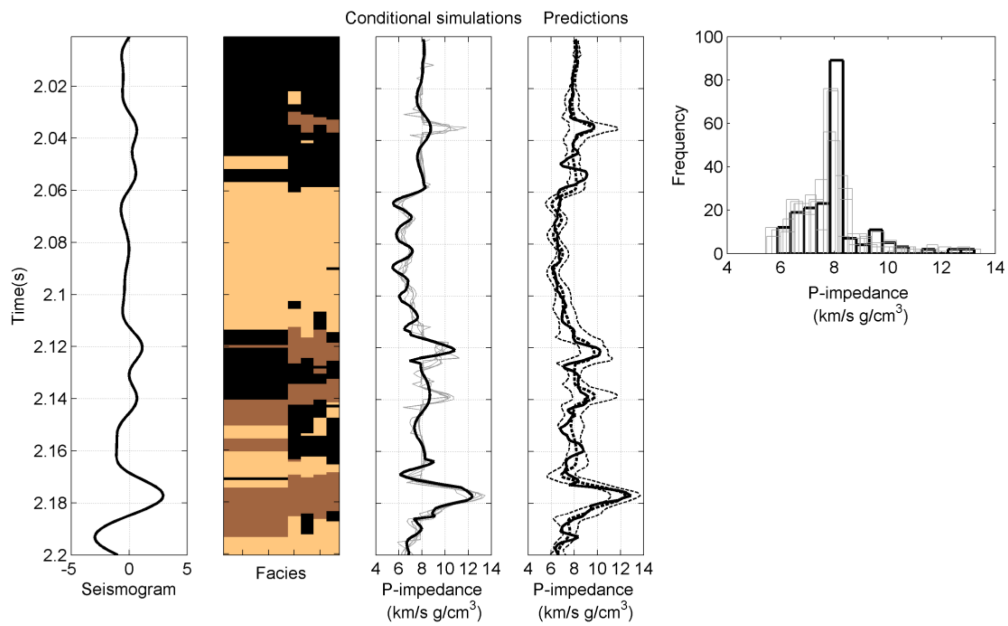

Figure 3 *Bayesian Gaussian mixture inversion of seismic data for P-impedance estimation, from left to right: actual seismogram (data **d** in Example 2); facies classification (the first column represents the actual log-facies profile, the other columns show 4 conditional simulations); P-impedance simulations (the black line shows to the first conditional simulation, grey lines represent four other simulations); predictions (the solid line represents the actual P-impedance log, dashed lines represent the MAP and the 90% prediction interval); and histograms of conditional simulations.*

Conclusions

We presented the analytical formulation of the Bayesian inverse problem under the assumptions of a Gaussian linear likelihood function and a Gaussian mixture random field of the prior model. This inversion approach allows estimating simultaneously the continuous model and the latent categorical variable. The analytical solution allows sampling from the posterior distribution with small computational cost.

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