# Convex Analysis, Sect. 3.1

## Exercise 1

Prove Proposition 3.1.1: A function $f : E \to (-\infty, +\infty]$ is sublinear iff it is positively homogeneous and subadditive.

**Proof: If:** $\forall x, y \in E, \lambda, \mu \in \mathbb{R}_+$:

$$f(\lambda x + \mu y) \underbrace{\leq f(\lambda x) + f(\mu y)}_{\text{subadditivity}} \underbrace{= \lambda f(x) + \mu f(y)}_{\text{positive homogeneity}}$$

**Only if:** Subadditivity is immediate (take $\lambda = \mu = 1$ in the definition of sublinearity). Positive homogeneity: first of all $f(0) = 0$, because

$$f(\vec{0}) = f(\vec{0}+\vec{0}) \leq f(\vec{0})+f(\vec{0}), \quad \text{and} \quad f(\vec{0}) = f(0\vec{0}+0\vec{0}) \leq 0f(\vec{0})+0f(\vec{0}) = 0,$$

and therefore $\forall x \in E : f(0x) = 0 = 0f(x)$. Finally, $\forall x \in E, \lambda \in \mathbb{R}_{++}$:

$$f(\lambda x) = f(\lambda x+0) \leq \lambda f(x)+f(0), \quad \text{and} \quad f(x) = f(\lambda^{-1}\lambda x+0) \leq \lambda^{-1}f(\lambda x)+f(0).$$

□

## Exercise 2

Consider $D = \{\, (x, y) \in \mathbb{R}^2 \mid y = 0 \text{ or } |y| \geq x^2 \,\}$. Prove that $0 \in \text{core } D \setminus \text{int } D$.

**Proof:** Let $(x_k, y_k) = (1/k, 1/k^3)$. Then $\lim_{k\to\infty}(x_k, y_k) = (0, 0)$ while $(x_k, y_k) \notin D$, for all $k \geq 2$. Therefore, $(0, 0) \notin \text{int } D$.

On the other hand, consider an arbitrary direction $(x, y) \in \mathbb{R}^2$. If $y = 0$ then $t(x, y) \in D$ for all $t \in \mathbb{R}$. If on the other hand $y \neq 0$ then for $0 < t < |y|/x^2$ (for all $t \in \mathbb{R}_+$ when $x = 0$) we have that $t|y| \geq (tx)^2$, showing that $0 \in\in \text{core } D$. □

## Exercise 3

Prove that the subdifferential is a closed convex set.

**Proof:**

$$\begin{aligned}\partial f(\bar{x}) &= \{\, \phi \in E \mid \langle \phi, x - \bar{x} \rangle \leq f(x) - f(\bar{x}), \forall x \in E \,\} \\ &= \cap_{x\in E}\{\, \phi \in E \mid \langle \phi, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) \,\}.\end{aligned}$$

Therefore, the subdifferential is an intersention of closed half-spaces, which are closed and convex sets, and therefore is closed and convex. □

## Exercise 4

Prove that for every set $C \subset E$ and every $\bar{x} \in C$, $\partial \delta_C(\bar{x}) = N_C(\bar{x})$.

**Proof:** $\phi \in \partial \delta_C(\bar{x}) \iff \forall x \in E : \langle \phi, x - \bar{x} \rangle \leq \delta_C(x) - \delta_C(\bar{x}) = \delta_C(x)$ $\iff \forall x \in C : \langle \phi, x - \bar{x} \rangle \leq \delta_C(x) - \delta_C(\bar{x}) = 0 \iff \phi \in N_C(\bar{x})$.

□

## Exercise 6

Prove Proposition 3.1.6: If the function $f : E \to (-\infty, +\infty]$ is convex and the point $\bar{x}$ lies in $\mathrm{dom} f$, then an element $\phi \in E$ is a subgradient of $f$ at $\bar{x}$ iff it satisfies $\langle \phi, \cdot \rangle \leq f'(\bar{x}, \cdot)$.

**Proof: If:** Assume that $\forall d \in E$: $\langle \phi, d \rangle \leq f'(\bar{x}, d)$ at some $\bar{x} \in \mathrm{dom} f$. Then, $\forall x \in E$:

$$\begin{aligned}\langle \phi, x - \bar{x} \rangle &\leq f'(\bar{x}, x - \bar{x}) \\ &= \underbrace{\lim_{t \downarrow 0} \frac{f(\bar{x} + t(x - \bar{x})) - f(\bar{x})}{t} \leq \left. \frac{f(\bar{x} + t(x - \bar{x})) - f(\bar{x})}{t} \right|_{t=1}}_{\text{Exercise 7, Section 2.1}} \\ &= f(x) - f(\bar{x}),\end{aligned}$$

and therefore $\phi \in \partial f(\bar{x})$. Note that Exercise 7 in Section 2.1 only deals with finite-valued convex functions. However the inequality remains true for extended real valued functions as well; indeed, if $f(x) = +\infty$, the inequality clearly remains valid.

**Only if:** For the sake of contradiction assume that $\langle \phi, x - \bar{x} \rangle > f'(\bar{x}, x - \bar{x}) + \epsilon$ for some $\phi, x \in E$, $\bar{x} \in \mathrm{dom} f$, and $\epsilon > 0$; in particular, $f'(\bar{x}, x - \bar{x}) < +\infty$. Then

$$\langle \phi, x - \bar{x} \rangle > f'(\bar{x}, x - \bar{x}) + \epsilon \underbrace{> \frac{f(\bar{x} + t(x - \bar{x})) - f(\bar{x})}{t}}_{\forall \text{ small } t > 0},$$

and as a result $\langle \phi, t(x - \bar{x}) \rangle > f(\bar{x} + t(x - \bar{x})) - f(\bar{x})$ for all small $t > 0$. Therefore, $\phi \notin \partial f(\bar{x})$. □

## Exercise 9

Prove $\partial \lambda_1(0) = \{ Y \in \mathcal{S}_+^n \mid \mathrm{tr} Y = 1 \}$.

**Proof:** For any $Y \in S_+^n$, $X \in S^n$ we have:

$$\langle Y, X - 0 \rangle = \underbrace{\langle Y, X \rangle \leq \lambda^T(Y)\lambda(X)}_{\text{Fan's inequality}} \leq \lambda_1(X) \sum_{i=1}^{n} \lambda_i(Y) = \mathrm{tr}(Y)[\lambda_1(X) - \underbrace{\lambda_1(0)}_{=0}].$$

Therefore, $\{ Y \in \mathcal{S}^n_+ \mid \mathrm{tr} Y = 1 \} \subset \partial\lambda_1(0)$.

On the other hand, fix $Y \in \partial\lambda_1(0) \subset \mathcal{S}^n$, and let $Y = U\mathrm{Diag}\lambda(Y)U^T$ be its spectral decomposition. Then

$$\pm\mathrm{tr}(Y) = \langle Y, \pm I\rangle = \langle Y, \pm I - 0\rangle \leq [\lambda_1(\pm I) - \underbrace{\lambda_1(0)}_{=0}] = \pm 1,$$

which implies that $\mathrm{tr} Y = 1$.

Further, let $X = U\mathrm{Diag}[\lambda(Y)]_- U^T$, where $[\lambda]_- = \min\{0, \lambda\}$ is the negative part of a number (we apply it componentwise to a vector). Since $X$ and $Y$ admit simultaneous ordered spectral decomposition, Fan's inequality holds with equality, and

$$0 \leq \sum_{i:\lambda_i<0} [\lambda_i(Y)]^2 = \lambda(Y)^T\lambda(X) = \langle Y, X\rangle = \langle Y, X-0\rangle \leq [\lambda_1(X)-\underbrace{\lambda_1(0)}_{=0}] \leq 0,$$

implying that $Y \in \mathcal{S}^n_+$. $\square$