# Convex Analysis, Sect. 2.2

## Exercise 1

Prove the implications $(ii) \implies (iii) \implies (i)$ in Theorem 2.2.6.

$(ii) \implies (iii)$:

**Proof:** If (2.2.2) is solvable, then $\forall x \in E$:

$$\left\langle \sum_{i=0}^{m} \lambda_i a^i, x \right\rangle = \sum_{i=0}^{m} \lambda_i \left\langle a^i, x \right\rangle = 0,$$

where $\lambda_i \geq 0$ and $\sum_{i=0}^{m} \lambda_i = 1$. Therefore the strict inequality $\langle a^i, x\rangle < 0$ (or $> 0$) for all $i$ is impossible, and (2.2.3) cannot be solvable. □

$(iii) \implies (i)$:

**Proof:** For the sake of contradiciton assume that $\log \sum_{i=0}^{m} \exp\langle a^i, x\rangle$ is unbounded from below. Then, for every $n \in \mathbb{N}$ there is $x_n \in E$ such that $\log \sum_{i=0}^{m} \exp\langle a^i, x_n\rangle < -n$ and $\sum_{i=0}^{m} \exp\langle a^i, x_n\rangle < \exp(-n)$. In particular $\exp\langle a^i, x_n\rangle < \exp(-n)$ for all $i = 0, \dots m$ since all the terms on the left are positive. Therefore $\langle a^i, x_n\rangle < -n < 0$ for all $i = 0, \dots m$ and (2.2.3) is solvable. □

## Exercise 3

Use the Basic separation theorem (2.1.6) to give another proof of Gordan's theorem.

**Proof:** If (2.2.2) is solvable then (2.2.3) is unsolvable, see $(ii) \implies (iii)$ in Exercise 1. Assume now that (2.2.2) is unsolvable, that is, $0 \notin S = \operatorname{conv}\{ a^1, \dots, a^m \}$. The set $S$ is convex (duh!) and closed (in fact even compact, see for example Exercise 5 (d)), so the point 0 can be separated from it. That is, there is $x \in E$ and $b \in \mathbb{R}$, such that $\langle 0, x\rangle > b \geq \langle a, x\rangle$ for all $a \in S$. In particular, $\langle a^i, x\rangle < 0$, meaning that $x$ solves (2.2.3). □

## Exercise 5

Suppose $\{ a^i \mid i \in I \}$ is a finite set of points in $E$. For any subset $J$ of $I$ define the cone

$$C_J = \left\{ \sum_{i \in J} \mu_i a^I \mid \mu_i \in \mathbb{R}_+ \right\}.$$

**(a)** Prove the cone $C_I$ is the union of those cones $C_J$, for which the set $\{ a^i \mid i \in J \}$ is linearly independent. Furthermore, prove directly that any such $C_J$ is closed.

**Proof:** Let us define two families of subsets: $\mathcal{J} = \{ J \subset I \mid$ vectors $a^i, i \in J$ are linearly independent $\}$ and $\mathcal{J}(x) = \{ J \subset I \mid x \in C_J \}$, where $x \in C_I$. For each $x \in C_I$, $\mathcal{J}(x)$ has only a finite number of elements (index sets). Let us now select an index set with the smallest number of indices in it (there could be several), and denote this subset by $J^*(x)$.

We claim that for every $x \in C_I$, $J^*(x) \in \mathcal{J}$. Indeed, for the sake of contradiction suppose that $\sum_{i \in J^*(x)} \beta_i a^i = 0$. Without any loss of generality we can assume that at least one $\beta_i$ is positive (otherwise we can multiply the previous equation with $-1$). Let us now select $i_0 \in J^*(x)$ such that $i_0 = \min_{i \in J^*(x)} \{\mu_i/\beta_i \mid \beta_i > 0\}$. Then

$$x = \sum_{i \in J^*(x)} \mu_i a^i = \sum_{i \in J^*(x) \setminus i_0} \underbrace{[\mu_i - (\mu_{i_0}/\beta_{i_0})\beta_i]}_{\geq 0, \text{ by constr. of } i_0} a^i \in C_{J^*(x) \setminus i_0}.$$

Therefore, $J^*(x) \setminus i_0 \in \mathcal{J}(x)$, which contradicts the construction of $J^*(x)$ as the set with the smallest number of elements in $\mathcal{J}(x)$.

Thus $C_I \subset \cup_{x \in C_I} C_{J^*(x)} \subset \cup_{J \in \mathcal{J}} C_J \subset C_I$, where the second inclusion is owing to the fact $\forall x \in C_I : J^*(x) \subset \mathcal{J}$ and the second inclusion is simply because $J \subset I \implies C_J \subset C_I$.

Now select an arbitrary $J \in \mathcal{J}$, and an arbitrary sequence of points $x^k \in C_J$ converging to a limit $\bar{x} \in E$. Let $\alpha^k \in \mathbb{R}^{|J|}$ be the "coordinates" of $x^k$ in the representation $x^k = \sum_{i \in J} \alpha_i^k a^i$. If $\|\alpha^k\|_{\mathbb{R}^{|J|}} \to \infty$ as $k \to \infty$ then the renormalized (bounded) sequence of coordinates $\alpha^k / \|\alpha^k\|_{\mathbb{R}^{|J|}}$ contains a convergent subsequence $k'$ with a limit $\hat{\alpha} \neq 0$ ($\|\hat{\alpha}\|_{\mathbb{R}^{|J|}} = 1$). This limit satisfies the equation $\sum_{i \in J} \hat{\alpha}_i a^i = \lim_{k' \to \infty} x^k / \|\alpha^k\| = 0$, which contradicts the linear independence of vectors $a^i$, $i \in J$.

Therefore the sequence of "coordinates" must be bounded in $\mathbb{R}^{|J|}$, thereby containing a convergent subsequence $k'$ with a limit $\bar{\alpha}$. Taking the limit along this subsequence results in the equality $\bar{x} = \sum_{i \in J} \bar{\alpha}_i a^i$ implying the inclusion $\bar{x} \in C_J$ since $\bar{\alpha}_i = \lim_{k' \to \infty} \alpha_i^{k'} \geq 0$ and therefore the closedness of $C_J$, $J \in \mathcal{J}$. □

**(b)** Deduce that any finitely generated cone is closed.

**Proof:** In view of **(a)** $C_I = \cup_{J \in \mathcal{J}} C_J$, that is, a finite union of closed sets. Therefore $C_I$ is also closed. □

**(c)** If the point $x$ lies in $\text{conv}\{ a^i \mid i \in I \}$, prove that in fact there is a subset $J \subset I$ of size at most $1 + \dim E$ such that $x$ lies in $\text{conv}\{ a^i \mid i \in J \}$

**Proof:** $x \in \text{conv}\{ a^i \mid i \in I \}$, if and only if $(x, 1) = \sum_{i \in I} \mu_i (a^i, 1)$, $\mu_i \geq 0$. In view of **(a)** we can assume that $(x, 1)$ can be represented using an index set $J$, such that the vectors $(a^i, 1)$, $i \in J$ are linearly independent. The cardinality of $J$ is therefore bounded from above by $\dim(E \times \mathbb{R}) = \dim E + 1$. $\square$

**(d)** Use part (c) to prove that if a subset of $E$ is compact then so is its convex hull.

**Proof:** One could for example argue as follows: let $x^k$ be a sequence in conv$S$, where $S \subset E$ is compact. By Caratheodory's thm. (i.e., part (c) of this exercise) $x^k = \sum_{i=1}^{n+1} \lambda_i^k a_i^k$, where $0 \leq \lambda_i^k \leq 1$, $\sum_{i=1}^{n+1} \lambda_i^k = 1$, $a_i^k \in S$. By switching to converging subsequences for all the terms, that is $\lambda_i^k \to \bar{\lambda}_i$, $a_i^k \to \bar{a}_i$, which can be done owing to the compactness of $S$ and that of the standard simplex in $\mathbb{R}^{n+1}$, we extract a convergent subsequence of $x^k \to \sum_{i=1}^{n+1} \bar{\lambda}_i \bar{a}_i \in \text{conv}S$. $\square$

## Exercise 6

Give another proof of the Farkas lemma by applying the Basic separation theorem (2.1.6) to the set defined by (2.2.11) and using the fact that any finitely generated cone is closed.

**Proof:** As before, if (2.2.8) admits a solution then (2.2.9) is impossible.

Let us now assume that (2.2.8) has no solutions. Then $c \notin C$, where $C = \{ \sum_{i=1}^m \mu_i a^i \mid \mu_i \geq 0 \}$. Clearly $C$ is convex, and also closed owing to Exercise 5 (b). Therefore $c$ and $C$ can be separated: that is, there is $x \in E$ and $b \in \mathbb{R}$ such that $\langle c, x \rangle > b \geq \langle a, x \rangle$, for all $a \in C$. Since $0 \in C$ it holds that $b \geq 0$. Finally, $b \geq \langle \mu_i a^i, x \rangle$ for all $\mu_i > 0$ is only possible when $\langle a^i, x \rangle \leq 0$, implying that (2.2.9) admits a solution $x$. $\square$