HINTS FOR SECTION 1.2

1. Prove $\mathbb{S}^n_+$ is a closed convex cone with interior $\mathbb{S}^n_{++}$.

This should be pretty straightforward.

2. Explain why $\mathbb{S}^2_+$ is not a polyhedron.

This as well.

7. The Fan and Cauchy–Schwarz inequalities.
   (a) For any matrices X in $\mathbb{S}^n$ and U in $\mathbb{O}^n$, prove $\|U^T XU\| = \|X\|$.
   (b) Prove the function λ is norm-preserving.
   (c) Explain why Fan's inequality is a refinement of the Cauchy-Schwarz inequality.

This one is pretty straightforward, too. Recall for part (c) that the Cauchy–Schwarz inequality holds on arbitrary Hilbert spaces (not only $\mathbb{R}^n$).

11. For a fixed volumn vector s in $\mathbb{R}^n$, define a linear map $A\colon \mathbb{S}^n \to \mathbb{R}^n$ by setting $AX = Xs$ for any matrix X in $\mathbb{S}^n$. Calculate the adjoint map A^*.

Recall that the adjoint A^* of A is the unique linear map $A^*\colon \mathbb{R}^n \to \mathbb{S}^n$ satisfying

$$\langle A^*x, X\rangle_{\mathbb{S}^n} = \langle x, AX\rangle_{\mathbb{R}^n}$$

for all $x \in \mathbb{R}^n$ and $X \in \mathbb{S}^n$. Thus it is necessary to find, for given $x \in \mathbb{R}^n$ and $X \in \mathbb{S}^n$, a *symmetric* matrix (A^*x) for which the equation above holds.

12* (Fan's inequality) For vectors x and y in $\mathbb{R}^n$ and a matrix U in $\mathbb{O}^n$, define

$$\alpha = \big\langle \operatorname{Diag} x, U^T(\operatorname{Diag} y)U \big\rangle.$$

   (a) Prove $\alpha = x^T Zy$ for some doubly stochastic matrix Z.
   (b) Use Birkhoff's theorem and Proposition 1.2.4 to deduce the inequality $\alpha \le [x]^T[y]$.
   (c) Deduce Fan's inequality (1.2.2).

Parts (b) and (c) are pretty straightforward. For part (a) it might be helpful to note that, for every fixed matrix U, the mapping $(x, y) \mapsto \alpha(x, y) := \langle \operatorname{Diag} x, U^T(\operatorname{Diag} y)U\rangle$ is bilinear, and thus there (necessarily) exists a unique matrix Z only depending on U such that $\alpha(x, y) = x^T Zy$. One 'only' has to find Z and show that it is doubly stochastic.

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