SELECTED HINTS, SECTION 4.3

1. (Weak duality) Prove that the primal and dual values p and d defined by equations (4.3.2) and (4.3.3) satisfy $p \geq d$.

This is a pretty straightforward calculation.

3. (Slater and compactness) Prove the Slater condition holds for problem (4.3.1) if and only if there is a point $\hat{x} \in \mathbb{E}$ for which the level sets
$$\{\lambda \in \mathbb{R}^m_+ : -L(\hat{x}; \lambda) \leq \alpha\}$$
are compact.

Note that the level sets of $L(\hat{x}; \cdot)$ are always closed (because L is continuous in λ). Thus compactness is equivalent to boundedness.

4. (Examples of duals) Lots and lots of them.

None of these examples is really difficult (although doing *all* of them probably takes rather long time). For the last problems it can be quite helpful to invoke the definition of the Fenchel conjugate at some point. Also, Proposition 3.3.3 (Log barriers) can be pretty useful for avoiding even messier calculations in (e) and the penalized semidefinite program. For the semidefinite program, you might want to recall the results of Section 1.2 and, in particular, Fan's (in)equality.

9. (Fenchel and Lagrangian duality) Let $\mathbb{Y}$ be a Euclidean space. By suitably rewriting the primal Fenchel problem
$$\inf_{x \in \mathbb{E}} \{f(x) + g(Ax)\}$$
for given functions $f : \mathbb{E} \to (-\infty, +\infty]$, $g : \mathbb{Y} \to (-\infty, +\infty]$, and linear $A : \mathbb{E} \to \mathbb{Y}$, interpret the dual Fenchel problem
$$\sup_{\phi \in \mathbb{Y}} \{-f^*(A^*\phi) - g^*(-\phi)\}$$
as a Lagrangian dual problem.

A good starting point might be, to introduce an auxiliary variable $y \in \mathbb{Y}$ and the condition $Ax = y$ into the given primal problem. Also, one can always split up the equality constraint $Ax = y$ into the two inequality constraints $Ax - y \leq 0$ and $y - Ax \leq 0$ (both to be interpreted componentwise).

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