SELECTED HINTS, SECTION 4.1

Throughout the book, one can find a strange misprint (or some very strange notation): The half-open interval $(-\infty, +\infty]$ is always printed as $(\infty, +\infty]$. Fortunately, the latter does not really make much sense, and thus this error is somehow easy to detect and correct.

1. (Examples of polars) Calculate the polars of the following sets:
   (a) $\operatorname{conv}(B \cup \{(1,1),(-1,-1)\}) \subset \mathbb{R}^2$.
   (b) $\left\{(x,y) \in \mathbb{R}^2 : y \geq b + \frac{x^2}{2}\right\} \quad (b \in \mathbb{R})$.

For part (a) note that we always have that $C^\circ = (\operatorname{conv} C)^\circ$ (the proof of this is straightforward). Thus you need not bother about the convex hull in the problem.

Part (b) is slightly trickier. One can start with determining the sign of the second component of elements in the polar of the given set. Then the problem basically becomes an exercise in the solution of inequalities. If one wants to get an impression about how the solution looks like, some basic knowledge of how to recognize different conic sections can be advantegeous.

5. (Polar sets) Suppose C is a nonempty subset of $\mathbb{E}$.
   (a) Prove $\gamma_C^* = \delta_{C^\circ}$.
   (b) Prove C° is a closed convex set containing 0.
   (c) Prove $C \subset C^{\circ\circ}$.
   (d) If C is a cone, prove $C^\circ = C^-$.
   (e) For a subset D of $\mathbb{E}$, prove $C \subset D$ implies $D^\circ \subset C^\circ$.
   (f) Prove C is counded if and only if $0 \in \operatorname{int} C^\circ$.
   (g) For any closed halfspace $H \subset \mathbb{E}$ containing 0, prove $H^{\circ\circ} = H$.
   (h) Prove Theorem 4.1.5 (Bipolar set).

This exercise appears to be rather long, but most of the sub-problems should not be too difficult.

One possibility for tackling (g) is to realize that any closed halfspace containing 0 can be written in the form $H = \{x : \langle x, s\rangle \leq t\}$ for some $s \in \mathbb{E}$ and $t \geq 0$.

For (h) it can help to recall (?) that whenever D is a closed convex set and $x \notin D$, then there exists a halfspace H with $D \subset H$, $x \notin H$. Or, alternatively, one can use the (equivalent) fact that the convex hull of a set C is the intersection of all closed halfspaces containing C.

13. (Existence of extreme points) Prove any nonempty compact convex set $C \subset \mathbb{E}$ has an extreme point, without using Minkowski's theorem, by considering the furthest point in C from the origin.

The hint is already given in the statement of the exercise.

14. Prove Lemma 4.1.7.

This should be pretty straightforward.

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