SELECTED HINTS, SECTION 3.2

Throughout the book, one can find a strange misprint (or some very strange notation): The half-open interval $(-\infty, +\infty]$ is always printed as $(\infty, +\infty]$. Fortunately, the latter does not really make much sense, and thus this error is somehow easy to detect and correct.

1. Prove the Lagrangian sufficient conditions (3.2.3).

The main point in the proof is not to use the convexity of either f or the functions g_i. In fact, the Lagrangian sufficient conditions hold without *any* assumptions concerning the extended real valued functions f and g_i. One only needs to employ the minimality assumption in the correct way.

4. For a matrix A in $\mathbb{S}^n_{++}$ and a real $b > 0$, use the Lagrangian sufficient conditions to solve the problem

$$\inf\bigl\{-\log\det X : \operatorname{tr} AX \le b,\ X \in \mathbb{S}^n_{++}\bigr\}.$$

You may use the fact that the objective function is convex with derivative $-X^{-1}$ (see Section 3.1, Exercise 21 (The log barrier)).

This is a straightforward computation. Note that the Lagrangian is convex and differentiable in X, and thus the optimality condition $0 = \nabla_X L(X, \lambda)$ is necessary and sufficient for a (global) minimiser.

6. (Extended convex function)
   (a) Give an example of a convex function that takes the values 0 and $-\infty$.
   (b) Prove the value function v defined by equation (3.2.4) is convex.
   (c) Prove that a function $h\colon \mathbb{E} \to [-\infty, \infty]$ is convex if and only if it satisfies the inequality

$$h(\lambda x + (1-\lambda)y) \le \lambda h(x) + (1-\lambda)h(y)$$

   for any points x and y in $\operatorname{dom} h$ (or $\mathbb{E}$ if h is proper) and any real λ in $(0, 1)$.
   (d) Prove that if the function $h\colon \mathbb{E} \to [-\infty, +\infty]$ is convex, then $\operatorname{dom}(h)$ is convex.

The main difficulty in this exercise is to apply all the definitions correctly:

- A function is called convex, if its epigraph is convex (the convexity of a set is defined in the "usual" manner).
- The epigraph of a function $f\colon \mathbb{E} \to [-\infty, +\infty]$ consists of all points $(x, \xi) \in \mathbb{E} \times \mathbb{R}$ such that $f(x) \le \xi$. Note here that the values of ξ are restricted to $\mathbb{R}$ and may not be infinite.
- The domain of a function $f\colon \mathbb{E} \to [-\infty, +\infty]$ consists of all points x where $f(x) < +\infty$. This implies in particular that all points x with $f(x) = -\infty$ are contained in $\operatorname{dom} f$.

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It might be a good idea to draw some sketches of convex *and non-convex* functions $f\colon \mathbb{R} \to [-\infty, +\infty]$ in order to get better used to the different concepts before starting this exercise.

7. (Nonexistence of multiplier) For the function $f\colon \mathbb{R} \to (-\infty, +\infty]$ defined by $f(x) = -\sqrt{x}$ for $x \in \mathbb{R}_+$ and $+\infty$ otherwise, show there is no Lagrange multiplier at the optimal solution of $\inf\bigl\{f(x) : x \leq 0\bigr\}$.

This exercise should be no problem at all.

11. (Normals to epigraphs) For a function $f\colon \mathbb{E} \to (-\infty, \infty]$ and a point $\bar{x}$ in $\operatorname{core}(\operatorname{dom} f)$, calculate the normal cone $N_{\operatorname{epi} f}(\bar{x}, f(\bar{x}))$.

Although it is not stated in the exercise, it makes sense to assume that the function f is convex, as in this case its epigraph is convex as well and we have defined normal cones only for convex sets. For non-convex functions (and thus non-convex epigraphs), there are different (sensible) definitions of normal cones, which all lead to different (sensible!) solutions of this problem.

In order to get an idea of how the solution might look like, it can be helpful to draw the epigraph of a convex function $f\colon \mathbb{R} \to \mathbb{R}$ and try to find the normal cone at points on the graph of f.