Convex Analysis, Sect. 3.3

Exercise 1

Use the definition + first order optimality conditions a lot. The same can be said for Exersises 2–3 as well.

Exercise 2

Convexity of $f_A(x) = x^T Ax/2$ can be for example inferred from the fact that $f_A = f_A^{**}$.

Exercise 3

Use the definition + first order optimality conditions a lot.

Exercise 4

In all these problems the supremum can only be attained at an extreme point of the set.

(e): when calculating $\delta_K^*(\phi)$ consider two cases: $\phi \in K^-$ and $\phi \notin K^-$.

Exercise 8

The inequality follows immediately from the definition of the conjugate function. The equality case requires some more work, but requires little beyond the definition of the subgradient.

Exercise 9

(a): Apply Fenchel–Young inequality to f and g at points involved in the primal and dual problems.

(b): Definition of the subgradient.

(c): Definition, direct computation.

(d): (3.3.9) implies that $\delta B \subset \text{dom } g - A\text{dom } f$ for some small ball $\delta B \in Y$.

(e): Apply the first part of the theorem to the perturbed optimization problem suggested in the book. Show that the primal/dual problems are solvable (and the solution to the primal problem is $\bar{x}$). Finally use the optimality conditions (these are proved in (f)).

(f): Return to Fenchel–Young inequalities in (a), which are satisfied exactly at the primal/dual optimality condition owing to the strong duality.

Exercise 23

Start from Theorem 3.3.5. You can benefit from the observation that $\delta_{b+K} = \delta_K(\cdot - b)$.