TMA4320 – Vedlegg til eksamen

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M1 Ikke-lineære ligninger

Intervallhalveringsalgoritmen

$f(x) = 0$

Input: a_0, b_0, tol, f
$k = 1$
while $(\frac{1}{2})^k(b_0 - a_0) > \text{tol}$
    $m_k = \frac{1}{2}(a_{k-1} + b_{k-1})$
    if $f(a_{k-1}) \cdot f(m_k) < 0$
        $b_k = m_k$; $a_k = a_{k-1}$
    else if $f(a_k) \cdot f(m_k) > 0$
        $b_k = b_{k-1}$; $a_k = m_k$
    else
        return($r = m_k$)
    end if
    k += 1
end while
return(m_k)

Newton's metode $f(x) = 0$

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)}}{f'(x^{(k)}}$$

Fikspunktiterasjon $x = g(x)$

$$x^{(k+1)} = g(x^{(k)})$$

M2 Interpolasjon

Lagrangeinterpolasjon

$$L_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$
$$P(x) = \sum_i y_i L_i(x)$$

Feilformel Lagrangeinterp av funksjon $f(x)$

$$e(x) = \frac{f^{n+1}(\xi)}{(n+1)!}(x - x_0)\cdots(x - x_n)$$

Dividerte differenser $f(x)$

$$f[x_i] = f(x_i),$$
$$f[x_i \cdots x_{i+k}] = \frac{f[x_{i+1} \cdots x_{i+k}] - f[x_i \cdots x_{i+k-1}]}{x_{i+k} - x_i}$$

Newton's form av interpolasjonspolynomet

$$P(x) = f[x_0] + \sum_{i=1}^{n} f[x_0 \cdots x_i] \prod_{j=0}^{i-1} (x - x_j)$$

Chebyshevpolynomer

$$T_k(x) = \cos(k \arccos x)$$

Rekursjonsformel

$$T_{k+1}(x) = 2x\, T_k(x) - T_{k-1}(x)$$

Nullpunkter i T_k

$$x_j = \cos(\frac{\pi}{2}\frac{2j-1}{k}), \; j = 1, \ldots, k.$$

M3 Numerisk integrasjon

Sammensatte formler ($h = \frac{b-a}{n}$)

Trapes

$$\int_a^b f(x)\,dx = \frac{h}{2} f(a) + h \sum_{k=1}^{n-1} f(a + kh) + \frac{h}{2} f(b) + E_n$$
$$E_n = -\frac{1}{12} h^2 f''(\xi)(b - a), \quad \xi \in (a, b)$$

Midtpunkt

$$\int_a^b f(x)\,dx = h \sum_{k=1}^{n} f\left(a + (k - \frac{1}{2})h\right) + E_n$$
$$E_n = \frac{1}{24} h^2 f''(\xi)(b - a), \quad \xi \in (a, b)$$

Simpson ($2mh = b - a, \; y_i = f(a + ih)$)

$$\int_a^b f(x)\,dx = \frac{h}{3}\left(y_0 + y_{2m} + 4\sum_{i=1}^{m} y_{2i-1} + 2\sum_{i=1}^{m-1} y_{2i}\right) + E_{2m}$$
$$E_{2m} = -\frac{1}{180} h^4 f^{(4)}(\xi)(b - a), \; \xi \in (a, b).$$

Adaptivt kvadratur

Feilestimat for trapesmetoden.

$$
\begin{aligned}
I &= \int_a^b f(x)dx \\
Q_{[a,b]}f &= \frac{b-a}{2}(f(a)+f(b)) \\
m &= \frac{a+b}{2} \\
I - Q_{[a,b]}f &\approx \frac{4}{3}(Q_{[a,m]} + Q_{[m,b]} - Q_{[a,b]})
\end{aligned}
$$

M4 Lineær algebra

Spesifikke vektornormer

$$
\begin{aligned}
\|x\|_1 &= \sum_{i=1}^{n} |x_i| \\
\|x\|_2 &= \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}} \\
\|x\|_\infty &= \max_{1\le i\le n} |x_i|
\end{aligned}
$$

Spesifikke matrisenormer

$$
\begin{aligned}
\|A\|_1 &= \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}| \\
\|A\|_2 &= \rho(A^T A)^{\frac{1}{2}} \\
\|A\|_\infty &= \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|
\end{aligned}
$$

Choleskyfaktorisering, $A = LL^T$

$$
\begin{aligned}
\ell_{ij} &= (a_{ij} - \sum_{k=1}^{j-1} \ell_{ik}\ell_{jk})/\ell_{jj},\ 1 \le j < i \\
\ell_{ii} &= \left(a_{ii} - \sum_{k=1}^{i-1} \ell_{ik}^2\right)^{\frac{1}{2}}
\end{aligned}
$$

Iterative metoder for $Ax = b$

$$
\begin{aligned}
a_{ii}x_i^{(k+1)} &= b_i - \sum_{j\ne i} a_{ij}x_j^{(k)} \text{ (Jacobi)} \\
a_{ii}x_i^{(k+1)} &= b_i - \sum_{j<i} a_{ij}x_j^{(k+1)} - \sum_{j>i} a_{ij}x_j^{(k)} \\
&\qquad\qquad \text{(Gauss-Seidel)} \\
a_{ii}x_i^{(k+1)} &= \omega b_i - \omega\sum_{j<i} a_{ij}x_j^{(k+1)} + (1-\omega)a_{ii}x_i^{(k)} \\
&\quad - \omega\sum_{j>i} a_{ij}x_j^{(k)} \text{ (SOR)}
\end{aligned}
$$

M5 Num løsn difflign

Generelle Runge-Kutta metoder

$$
\begin{aligned}
k_i &= f(t_n + c_i h, y_n + h\sum_j a_{ij}k_j) \\
y_{n+1} &= y_n + h\sum_i b_i k_i
\end{aligned}
$$

Kuttas fjerde ordens metode

$$
\begin{aligned}
k_1 &= f(t_n, y_n) \\
k_2 &= f(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1) \\
k_3 &= f(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2) \\
k_4 &= f(t_n + h, y_n + hk_3) \\
y_{n+1} &= y_n + h(\frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4)
\end{aligned}
$$

Forbedret Eulers metode (eksplisitt trapes)

$$
\begin{aligned}
k_1 &= f(t_n, y_n) \\
k_2 &= f(t_n + h, y_n + hk_1) \\
y_{n+1} &= y_n + \frac{h}{2}(k_1 + k_2)
\end{aligned}
$$

Implisitte metoder

$$
\begin{aligned}
y_{n+1} &= y_n + hf(t_{n+1}, y_{n+1}) \text{ (Baklengs Euler)} \\
y_{n+1} &= y_n + \frac{h}{2}\big(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\big) \text{ (Trapes)} \\
y_{n+1} &= y_n + hf(t_n + \frac{1}{2}h, \frac{1}{2}(y_n + y_{n+1})) \text{ (Midtpunkt)}
\end{aligned}
$$

Ordensbetingelser for Runge–Kutta metoder

$p = 1$	$\sum_i b_i = 1$
$p = 2$	$\sum_i b_i c_i = \frac{1}{2}$
$p = 3$	$\sum_i b_i c_i^2 = \frac{1}{3}$
	$\sum_{i,j} b_i a_{ij} c_j = \frac{1}{6}$
$p = 4$	$\sum_i b_i c_i^3 = \frac{1}{4}$
	$\sum_{i,j} b_i c_i a_{ij} c_j = \frac{1}{8}$
	$\sum_{i,j} b_i a_{ij} c_j^2 = \frac{1}{12}$
	$\sum_{i,j,k} b_i a_{ij} a_{jk} c_k = \frac{1}{24}$

Stabilitet av Runge–Kutta metoder.

Testligning

$$y' = \lambda y, \quad \lambda \in \mathbb{C}$$

Stabilitetsfunksjon: Rasjonal funksjon $R(z)$ slik at

$$y_{n+1} = R(h\lambda)y_n$$

Stabilitetsområde

$$S_R = \{z \in \mathbb{C} : |R(z)| \le 1\}$$