NTNU
Kunnskap for en bedre verden

Institutt for matematiske fag

Eksamensoppgave i
TMA4320 Introduksjon til vitenskapelige beregninger

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Eksamenstid (fra–til): 09:00–13:00
Hjelpemiddelkode/Tillatte hjelpemidler: B: Spesifiserte trykte hjelpemidler tillatt:

- K. Rottmann: Matematisk formelsamling

Bestemt, enkel kalkulator tillatt.

Målform/språk: bokmål
Antall sider: 6
Antall sider vedlegg: 0

Kontrollert av:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
sort/hvit ☒	**farger** ☐
skal ha flervalgskjema ☐	

Dato Sign

Oppgave 1

a) Vi betrakter en likning

$$\tan(x) = 1, \qquad x \in (-\pi/2, \pi/2). \tag{1}$$

Vi vet at minst en rot x tilhører intervallet $(0, 1)$ fordi $\tan(0) = 0$ og $\tan(1) \approx 1.6$. Bruk halveringsmetoden til å finne en numerisk tilnærming $\hat{x}$ til denne roten x slik at forskjellen $|\hat{x} - x|$ er garantert mindre enn 0.1. (Vi antar at roten er ukjent, ellers trenger vi ikke å løse likningen.)

Solution: The initial interval $(0, 1)$ contains at least one root as $\tan(0)-1 < 0$ and $\tan(1) - 1 > 0$; since the function $\tan(x) - 1$ is continuous on this interval it must assume value 0 somewhere. The center of the interval 0.5 is no further than $(1 - 0)/2 = 0.5$ from the root, therefore we continue with bisection until the accuracy requirement is satisfied.

The table below explains the iteration history.

Iteration	x_{left}	x_{right}	x_{center}	$\tan(x_{\text{center}}) - 1$	$(x_{\text{right}} - x_{\text{left}})/2$
1	0	1	0.5	-0.4537	0.5
2	0.5	1	0.75	-0.0684	0.25
3	0.75	1	0.8750	0.1974	0.125
4	0.75	0.875	0.8125	0.0557	0.0625

Thus the found numerical approximation is 0.8125, which is guaranteed to be no further than $0.0625 < 0.1$ from the actual root $\pi/4$. Note that $|0.8125 - \pi/4| \approx 0.0271 < 0.1$.

Oppgave 2

a) En Euler–Bernoulli bjelke (som dere har sett i prosjekt 2) kan betraktes matematisk som en kurve $q(x)$ som oppfyller en differensiallikning

$$q''''(x) = 0, \tag{2}$$

med passende randbetingelsene.

Hvor mange muligheter finnes det for bjelker som passerer gjennom punktene $(0, 1)$, $(2, 2)$, $(3, -2)$? Beskriv funksjonen $q(x)$ for slike bjelker, og gi et specifikk eksempel.

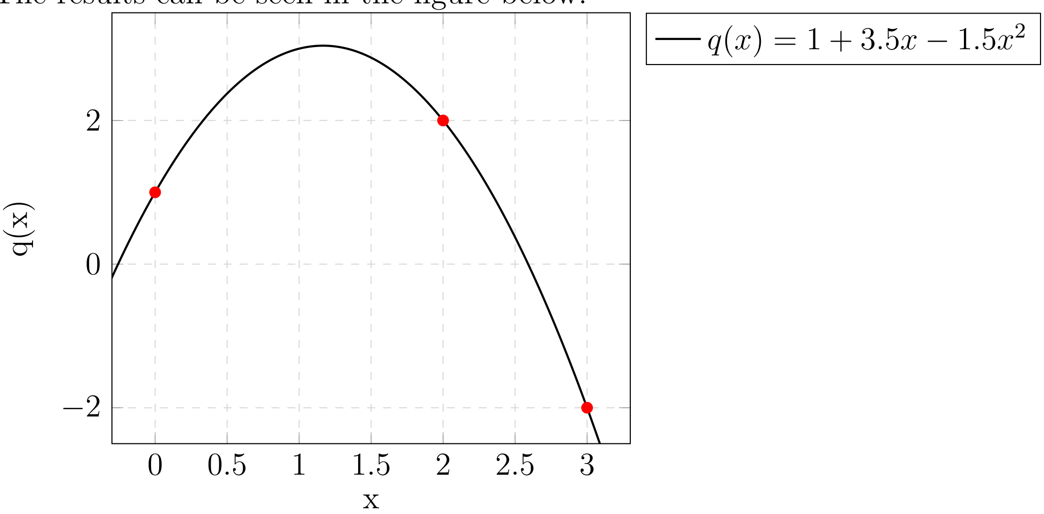
Solution: Cubic polynomials satisfy the differential equation, and there are infinitely many polynomials of degree ≥ 3 passing through given three points.

Therefore the answer to the first question is: $q(x)$ is a cubic polynomial and there are infinitely many beams passing through the given points.

For a specific example we could use polynomial interpolation. For example, Newton's divided differences yield: $q[0] = 1$, $q[2] = 2$, $q[3] = -2$, $q[0,2] = 0.5$, $q[2,3] = -4$, $q[0,3] = -1.5$, and the final polynomial is

$$q(x) = 1 + 0.5(x - 0) - 1.5(x - 0)(x - 2) = 1 + 3.5x - 1.5x^2$$

The results can be seen in the figure below:



Polynomet $P_n(x)$ av lavest mulig grad som interpolerer en glatt funksjon $F(x)$ i punktene $x_1, \ldots, x_n$ oppfyller følgende feilestimat:

$$F(x) - P_n(x) = \frac{(x - x_1)\ldots(x - x_n)}{n!} F^{(n)}(c), \tag{3}$$

med $c \in [\min\{x, x_1, \ldots, x_n\}, \max\{x, x_1, \ldots, x_n\}]$.

b) Vi betrakter en funksjon $F(x) = \cos(x)$ på intervallet $[0, 2]$ og interpolerer den i punktene $0 \leq x_1 < \cdots < x_n \leq 2$ ved hjelp av et polynom P_n av lavest mulig grad. Finn n slik at feilen $e = \max_{x\in[0,2]} |F(x) - P_n(x)| < 0.1$ uansett hvordan punktene $x_1, \ldots, x_n$ plasseres på intervallet.

Solution: Let us first estimate the right hand side of (3) from above: $|F^{(n)}(c)| \leq 1$ for all n and c; $|x - x_i| \leq 2$ for all $x, x_i \in [0, 2]$. Therefore we have the estimate $e \leq 2^n/n!$. We can see how this estimate behaves as a function of n:

n	1	2	3	4	5	6
$2^n/n!$	2	2	1.33	0.67	0.27	0.09

Thus for $n = 6$ (or, as a matter of fact, larger) the largest interpolation error will be smaller than 0.1 as required.

Oppgave 3

a) Beregn en tilnærmelse til integralet

$$i = \int_0^2 \frac{\mathrm{d}x}{x^{1/3}}$$

ved hjelp av midtpunktregelen basert på 1 og 2 "paneler".

Solution: Direct computation. For one panel we have:

$$i \approx 2\frac{1}{1^{1/3}} = 2,$$

and for two panels we get

$$i \approx 1\left[\frac{1}{0.5^{1/3}} + \frac{1}{1.5^{1/3}}\right] \approx 2.1335.$$

Note that the exact value is $i = 1.5x^{2/3}|_{x=0}^{x=2} \approx 2.3811$.

b) Vi vil tilnærme et integral $\int_0^1 f(x)\,\mathrm{d}x$ ved hjelp av et numerisk kvadratur $Q_{[0,1]}f = w_1 f(0.25) + w_2 f(0.75)$. Bestem vektene w_i slik at kvadraturet $Q_{[0,1]}f$ har høyest mulig presisjonsgrad[1] på intervallet $[0, 1]$. Rapporter presisjonsgraden du har funnet.

Solution: For polynomials of degree 0 we get

$$Q_{[0,1]}c = c(w_1 + w_2) = \int_0^1 c\,\mathrm{d}x = c,$$

for an arbitrary constant c. Therefore $w_1 + w_2 = 1$ if we want the quadrature to be exact for polynomials of degree 0.

Note that the value of the constant c does not participate in the equation since both the quadrature and the integral are linear with respect to the

[1]Høyest mulig grad av et vilkårlig polynom $p(x)$ som integreres uten feil, dvs $\int_0^1 p(x)\,\mathrm{d}x = Q_{[0,1]}p$

integrand f. Thus for polynomials of degree 1 it is sufficient to test for example $p(x) = x$:

$$Q_{[0,1]}x = 0.25w_1 + 0.75w_2 = \int_0^1 x \,\mathrm{d}x = 0.5.$$

We thus get a system of two linear algebraic equations with two unknowns, which can be solved to give $w_1 = w_2 = 0.5$.

We can check that

$$Q_{[0,1]}x^2 = 0.5(0.25^2 + 0.75^2) \neq \int_0^1 x^2 \,\mathrm{d}x = 1/3,$$

and therefore quadrature is not exact for polynomials of degree two and its degree of precision is 1.

Oppgave 4 Vi betrakter to ladninger (i en dimensjon) med posisjoner $x_1(t)$ og $x_2(t)$ i tidspunkt $t > 0$. Ladningenes posisjon oppfyller et system av to differensiallikninger (Coulombs + Newtons lover):

$$\begin{aligned} x_1''(t) &= -\frac{1}{(x_1(t) - x_2(t))^2}, \\ x_2''(t) &= \frac{1}{(x_1(t) - x_2(t))^2}, \end{aligned} \tag{4}$$

med begynnelsesbetingelsene $x_1(0) = 0$, $x_1'(0) = 1$, $x_2(0) = 1$, $x_2'(0) = 0$.

a) Skriv om (4) til et system av førsteordens differensiallikninger.

Solution: Let us put $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1'$, $y_4 = x_2'$. Then

$$y'(t) = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}'(t) = \begin{pmatrix} y_3(t) \\ y_4(t) \\ -(y_1(t) - y_2(t))^{-2} \\ (y_1(t) - y_2(t))^{-2} \end{pmatrix} =: F(y(t)), \tag{5}$$

The initial conditions are $y(0) = [0, 1, 1, 0]^{\mathrm{T}}$.

b) Gjør et steg med den *eksplisitte trapesmetoden* (Heuns metode) for systemet funnet i **a)** med tidsdiskretisering $h = 0.1$.

Solution: The method is a two stage Runge–Kutta method. We compute:

$$\begin{aligned}
w_0 &= y(0) = [0,1,1,0]^{\mathrm{T}} \\
k_1 &= F(w_0) = [1,0,-1,1]^{\mathrm{T}} \\
k_2 &= F(w_0 + hk_1) = F(0.1,1,0.9,0.1) \approx [0.9,0.1,-1.2346,1.2346]^{\mathrm{T}} \\
w_1 &= w_0 + h/2(k_1+k_2) \approx [0.0950,1.0050,0.8883,0.1117]^{\mathrm{T}}
\end{aligned}$$

Oppgave 5

a) Beregn Cholesky–faktoriseringen av matrisen

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 20 \end{pmatrix}$$

Solution: One could either follow the algorithm from the textbook or derive it from scratch for this small matrix. Let

$$L = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix}$$

Then

$$LL^{\mathrm{T}} = \begin{pmatrix} L_{11}^2 & L_{11}L_{21} \\ L_{11}L_{21} & L_{21}^2 + L_{22}^2 \end{pmatrix}$$

Thus $L_{11} = A_{11}^{1/2} = 1$, $L_{21} = A_{21}/L_{11} = 2/1 = 2$, and finally $L_{22} = (A_{22} - L_{21}^2)^{1/2} = (20-2^2)^{1/2} = 4$.

It is always a good idea to verify the computation: $A = LL^{\mathrm{T}}$.

b) Ved hjelp av beregningene i **a)**, bestem $x \in \mathbb{R}^2$ slik at $Ax = b = [1,-14]^{\mathrm{T}}$.

Solution: Let $Ly = b$; then $y_1 = b_1 = 1$ and $2y_1 + 4y_2 = -14$, or $y_2 = -4$.

It remains to solve the system $L^{\mathrm{T}}x = y$. $4x_2 = -4$ and therefore $x_2 = -1$. Then $x_1 + 2x_2 = 1$ or $x_1 = 3$.

It is always a good idea to verify the computation: $Ax = b$.

c) Med A og b som i **a)** og **b)**, gjør en iterasjon med Gauss–Seidel metoden med startverdi $x^{(0)} = [1,2]$.

Solution: Gauss–Seidel is a particular case of matrix splitting iterative algorithms of the type

$$Mx^{(k+1)} - Nx^{(k)} = b,$$

where $A = M - N$. In Gauss–Seidel method M is chosen to be either upper or lower triangular part of the matrix. Here we will use the lower triangular matrix.

Thus

$$\begin{pmatrix} 1 & 0 \\ 2 & 20 \end{pmatrix} x^{(1)} = b - \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} x^{(0)} = \begin{pmatrix} -3 \\ -14 \end{pmatrix}$$

This triangular system is easily solvable (which is the point with matrix splitting algorithms): $x_1^{(1)} = -3$, $2x_1^{(1)} + 20x_2^{(1)} = -14$, or $x_2^{(1)} = (-14 + 2 \cdot 3)/20 = -0.4$.

Arguably in this case it is better to choose M to the the upper triangular part of A, which results in a much better approximation (after one iteration) $x^{(1)} = [2.6, -0.8]^{\mathrm{T}}$.