Solutions_3

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1 Exercise 5.2.3

Apply composite Simpson's rule with $m = 1, 2, 4$ panels to approximate the integrals:

$$\text{(a)} \quad \int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}, \quad \text{(b)} \quad \int_0^{\pi/2} \cos(x)\,\mathrm{d}x = 1, \quad \text{(c)} \quad \int_0^1 e^x\,\mathrm{d}x = e - 1,$$

and report the errors.

Solution

(a) $f(x) = x^2$.

For $m = 1$ we put $h = (1-0)/2 = 1/2$, $x_0 = 0$, $x_1 = 1/2$, $x_2 = 1$; $y_0 = f(x_0) = 0$, $y_1 = f(x_1) = 1/4$, $y_2 = f(x_2) = 1$. The approximation is then $I \approx h/3(y_0 + 4y_1 + y_2) = (0 + 4 * 1/4 + 1)/6 = 1/3$. The error is thus 0.

For $m = 2$ we have $h = (1-0)/4 = 1/4$, $(x_0, x_1, \ldots, x_4) = (0, 1/4, 1/2, 3/4, 1)$, $(y_0, y_1, \ldots, y_4) = (0, 1/16, 1/4, 9/16, 1)$. The approximation is then $I \approx h/3(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) = (0 + 4 \cdot 1/16 + 2 \cdot 1/4 + 4 \cdot 9/16 + 1)/12 = (1/4 + 2/4 + 9/4 + 4/4)/12 = 16/(3 \cdot 4 \cdot 4) = 1/3$. The error is thus 0.

One could continue with $m = 3$, but the error has to be zero. Indeed, Simpson's rule is based on quadratic interpolation polynomials, which means that x^2 will be represented exactly by the interpolating polynomial and the quadrature will be exact.

(b) $f(x) = \cos(x)$

For $m = 1$ we put $h = (\pi/2 - 0)/2 = \pi/4$, $x_0 = 0$, $x_1 = \pi/4$, $x_2 = \pi/2$; $y_0 = f(x_0) = 1$, $y_1 = f(x_1) = 1/\sqrt{2}$, $y_2 = f(x_2) = 0$. The approximation is then $I \approx h/3(y_0 + 4y_1 + y_2) = (1 + 4/\sqrt{2} + 0) \cdot \pi/12 \approx 1.00227$. The error is thus ≈ 0.00227.

For $m = 2$, $h = \pi/8$, $(x_0, \ldots, x_4) = (0, \pi/8, \pi/4, 3\pi/8, \pi/2)$, $(y_0, \ldots, y_4) \approx (1, 0.9238795, 1/\sqrt{2}, 0.38268343, 0)$. $I \approx h/3(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) \approx 1.000135$, giving the error ≈ 0.000135.

For $m = 3$, $h = \pi/6$, $I \approx 1.00002631$, errror ≈ 0.00002631.

(c) $f(x) = \exp(x)$

For $m = 1$ we put $h = (1-0)/2 = 1/2$, $x_0 = 0$, $x_1 = 1/2$, $x_2 = 1$; $y_0 = f(x_0) = 1$, $y_1 = f(x_1) = e^{1/2}$, $y_2 = f(x_2) = e$. The approximation is then $I \approx h/3(y_0 + 4y_1 + y_2) = (1 + 4\sqrt{e} + e)/6 \approx 1.71886115$. The error is thus $\approx 1.71886115 - e + 1 \approx 5.79 \cdot 10^{-4}$.

For $m = 2$, $h = 1/4$, $(x_0, \ldots, x_4) = (0, 1/4, 1/2, 3/4, 1)$, $\ldots I \approx h/3(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) \approx 1.71831884192175$, giving the error $\approx 3.70 \cdot 10^{-5}$.

For $m = 3$, $h = 1/6$, $I \approx 1.71828916992083$, errror $\approx 7.34 \cdot 10^{-6}$.

2 Exercise 5.2.11

Find the degree of precision of the following approximation for $\int_{-1}^{1} f(x)\,\mathrm{d}x$:

First of all, let us evaluate the integral of a monomial x^n, $n \geq 0$:

$$\int_{-1}^{1} x^n \, \mathrm{d}x = \left[\frac{x^{n+1}}{n+1}\right]_{x=-1}^{x=1} = \begin{cases} 0, & \text{for odd } n, \\ 2/(n+1), & \text{for even } n. \end{cases}$$

(a) $f(1) + f(-1)$:
Let us apply the quadrature to the monomial x^n, $n \geq 0$:

$$1^n + (-1)^n = \begin{cases} 0, & \text{for odd } n, \\ 2, & \text{for even } n, \end{cases}$$

Therefore the quadrature agrees with the integral for $n = 0, 1$ (and all odd n), and its degree of precision is 1. (One could also observe that this is trapezoid quadrature on $[-1, 1]$, which is known to be exact for polynomials up to degree 1.)

(b) 2/3[f(-1) + f(0) + f(1)]. We check the quadrature on monomials of increasing degree:

$$2/3[(-1)^n + 0^n + (-1)^n] = \begin{cases} 2, & \text{for } n = 0, \\ 0, & \text{for odd } n, \\ 4/3, & \text{for even } n > 0, \end{cases}$$

which agrees with the integral for $n = 0, 1$ (and odd n). Thus degree of precision is 1.

(c) We check the quadrature on monomials of increasing degree:

$$[(-1/\sqrt{3})^n + (1/\sqrt{3})^n] = \begin{cases} 0, & \text{for odd } n, \\ 2/3^{n/2}, & \text{for even } n, \end{cases}$$

which agrees with the integral for $n = 0, 1, 2, 3$ (and odd n). Thus degree of precision is 3. (This is an example of Gaussian quadrature, which are exact for polynomials of degree up to $2n-1$ when n points are used.)

3 Exercise 5.2.16

Use the fact that the error term of Boole's Rule (see Exercise 5.2.15) is proportional to $f^{(6)}(c)$ to find the exact error term.

Solution

We apply the quadrature to the monomial x^6, for which $f^{(6)}(c) \equiv 6!$, for all c. Thus

$$\int_0^{4h} x^6 \, \mathrm{d}x = \left[\frac{x^7}{7}\right]_{x=0}^{x=4h} = \frac{4^7 h^7}{7} = \frac{2h}{45}(7 \cdot 0^6 + 32 \cdot h^6 + 12 \cdot (2h)^6 + 32 \cdot (3h)^6 + 7 \cdot (4h)^6) + C6!$$

and we are interested in finding C.

```
In [1]: from sympy import *
        init_printing()
        h,C=symbols('h C')
        integral = (4*h)**7/7
        quadrature = 2*h/45*(32*h**6 + 12*(2*h)**6 + 32*(3*h)**6 + 7*(4*h)**6)
        simplify(integral-quadrature)
```

Out[1]:

$$-\frac{128h^7}{21}$$

Thus $C = -128h^7/(21 \cdot 6!) = -8h^7/945$

4 Exercise 5.2.12

Wee need to find c_1, c_2, c_3 such that the rule

$$\int_0^1 f(x)\,\mathrm{d}x \approx c_1 f(0) + c_2 f(0.5) + c_3 f(1)$$

has degree of precision greater than one.

To have degree of precision 0 we need that

$$\int_0^1 1\,\mathrm{d}x = 1 = c_1 f(0) + c_2 f(0.5) + c_3 f(1) = c_1 + c_2 + c_3.$$

Similarly, for degree of precision 1 we need

$$\int_0^1 x\,\mathrm{d}x = 1/2 = c_1 f(0) + c_2 f(0.5) + c_3 f(1) = 0.5c_2 + c_3.$$

Finally, for degree of precision 2 we need

$$\int_0^1 x^2\,\mathrm{d}x = 1/3 = c_1 f(0) + c_2 f(0.5) + c_3 f(1) = 0.25c_2 + c_3.$$

This gives us 3 linear equations in 3 unknowns, whose solution is $c_1 = c_3 = 1/6$, $c_2 = 2/3$. This is the same as Simpson's rule.

5 Exercise 3b), Exam 08.2016

La $M_{[a,b]}f$ og $T_{[a,b]}f$ være midpunkt og trapezoid kvadraturer med $n = 1$ panel for funksjonen f på interval $[a, b]$. Feilestimatetene for disse kvadraturer er gitt av

$$\int_a^b f(x)\,\mathrm{d}x = M_{[a,b]}f + \frac{h^3}{24}f''(c) + O(h^4), \qquad \text{og}$$
$$\int_a^b f(x)\,\mathrm{d}x = T_{[a,b]}f - \frac{h^3}{12}f''(c) + O(h^4),$$

hvor $c = (a + b)/2$, og $h = b - a$.

La oss definere en ny kvadratur som $Q_{[a,b]}f = \alpha M_{[a,b]}f + \beta T_{[a,b]}f$. Bestem verdiene α, β slik at

$$\int_a^b f(x)\,\mathrm{d}x = Q_{[a,b]}f + O(h^4).$$

Solution: We have

$$M_{[a,b]}f = \int_a^b f(x)\,\mathrm{d}x - \frac{h^3}{24}f''(c) + O(h^4), \qquad \text{og}$$

$$T_{[a,b]}f = \int_a^b f(x)\,\mathrm{d}x + \frac{h^3}{12}f''(c) + O(h^4),$$

and therefore

$$Q_{[a,b]}f = (\alpha+\beta)\int_a^b f(x)\,\mathrm{d}x + (2\beta-\alpha)\frac{h^3}{24} + O(h^4).$$

As a result we get a system of equations

$$\alpha + \beta = 1,$$
$$2\beta - \alpha = 0.$$

Thus $\alpha = 2/3$, $\beta = 1/3$, which in fact gives us Simpson's rule:

$$\frac{2}{3}M_{[a,b]}f + \frac{1}{3}T_{[a,b]}f = \frac{2h}{3}f(c) + \frac{h}{6}(f(a)+f(b)) = \frac{h}{6}[f(a)+4f(c)+f(b)].$$

6 Exercise 5.4.1

Apply Adaptive Quadrature by hand, using the Trapezoid Rule with tolerance $TOL = 0.05$ to approximate the integrals. Find the approximation error.

Solution

(a) In this case, $f(x) = x^2$, $a_0 = 0$, $b_0 = 1$.

We begin with $n = 1$ interval, $a = a_0$, $b = b_0$. We will use $S[a,b]$ to denote the Trapezoid quadrature applied on the interval $[a,b]$.

$c = (a+b)/2$, $S[a,b] = (0^2+1^2)/2 = 1/2$, $S[a,c] = (0^2+(1/2)^2)/4 = 1/16$, $S[c,b] = ((1/2)^2+1^2)/4 = 5/16$. $|S[a,b]-S[a,c]-S[c,b]| = 1/8 = 0.125 < 3\cdot 0.05\cdot((b-a)/(b_0-a_0)) = 0.15$. Thus we stop with the approximation $S[a,c]+S[c,b] = 3/8 = 0.375$ and an error estimate $-(S[a,b]-S[a,c]-S[c,b])/3 = -1/24 \approx -0.0417$.

The actual integration error is $\int_0^1 x^2 - 3/8 = 1/3 - 3/8 = -1/24$. Our estimate is exact because f'' is constant in this case, thus the approximate equation (5.38) in the book is actually exact.

(b) $f(x) = \cos(x)$, $a_0 = 0$, $b_0 = \pi/2$.

$S[0,\pi/2] = (\pi/2)\cdot(1+0)/2 = \pi/2 \approx 1.5708$, $S[0,\pi/4] = (\pi/4)\cdot(1+1/\sqrt{2})/2 \approx 0.7854$, $S[\pi/4,\pi,2] = (\pi/4)\cdot(1/\sqrt{2}+0)/2 \approx 0.27768$. $|S[a,b]-S[a,c]-S[c,b]| \approx 0.50772 \geq 3\cdot 0.05\cdot(\pi/2)/(\pi/2) = 0.15$.

Thus we need to split the interval and apply the adaptive algorithm recursively.

$S[0,\pi/4] = (\pi/4)\cdot(1+1/\sqrt{2})/2 \approx 0.7854$, $S[0,\pi/8] = (\pi/8)\cdot(1+\cos(\pi/8))/2 \approx 0.37775$, $S[\pi/8,\pi/4] = (\pi/8)\cdot(\cos(\pi/8)+\cos(\pi/4))/2 \approx 0.32024$, $|S[0,\pi/4]-S[0,\pi/8]-S[\pi/8,\pi/4]| \approx 0.087410 \geq 3\cdot 0.05\cdot(\pi/4)/(\pi/2) = 0.075$.

Thus we need to split this interval again apply the adaptive algorithm recursively.

$S[0,\pi/8] = (\pi/8)\cdot(1+\cos(\pi/8))/2 \approx 0.37775$, $S[0,\pi/16] = (\pi/16)\cdot(1+\cos(\pi/16))/2 \approx 0.19446$, $S[\pi/16,\pi/8] = (\pi/16)\cdot(\cos(\pi/16)+\cos(\pi/8))/2 \approx 0.18699$, $|S[0,\pi/8]-S[0,\pi/16]-S[\pi/16,\pi/8]| \approx 0.0037 < 3\cdot 0.05\cdot(\pi/8)/(\pi/2) = 0.0375$.

Thus we are done on the interval $[0,\pi/8]$ with the approximation $S[0,\pi/16]+S[\pi/16,\pi/8] \approx 0.38145$.

Let us look at the interval $[\pi/8, \pi/4]$ now. $S[\pi/8, \pi/4] = (\pi/8) \cdot (\cos(\pi/8) + \cos(pi/4))/2 \approx 0.32024$, $S[\pi/8, 3\pi/16] = (\pi/16) \cdot (\cos(\pi/8) + \cos(3\pi/16))/2 \approx 0.17233$, $S[3\pi/16, \pi/4] = (\pi/16) \cdot (\cos(3\pi/16) + \cos(\pi/4))/2 \approx 0.15105$, $|S[\pi/8, \pi/4] - S[\pi/8, 3\pi/16] - S[3\pi/16, \pi/4]| \approx 0.00314 < 3 \cdot 0.05 \cdot (\pi/8)/(\pi/2) = 0.0375$.

Thus we are done on the interval $[\pi/8, \pi/4]$ with the approximation $S[\pi/8, 3\pi/16] + S[3\pi/16, \pi/4] \approx 0.32338$.

Let us look at the interval $[\pi/4, \pi/2]$ now. $S[\pi/4, \pi/2] = (\pi/4) \cdot (\cos(\pi/4) + \cos(pi/2))/2 \approx 0.27768$, $S[\pi/4, 3\pi/8] = (\pi/8) \cdot (\cos(\pi/4) + \cos(3\pi/8))/2 \approx 0.21398$, $S[3\pi/8, \pi/2] = (\pi/8) \cdot (\cos(3\pi/8) + \cos(\pi/2))/2 \approx 0.075140$, $|S[\pi/4, \pi/2] - S[\pi/4, 3\pi/8] - S[3\pi/8, \pi/2]| \approx 0.01144 < 3 \cdot 0.05 \cdot (\pi/4)/(\pi/2) = 0.075$.

Thus we are done on the interval $[\pi/4, \pi/2]$ with the approximation $S[\pi/4, 3\pi/8] + S[3\pi/8, \pi/2] \approx 0.28912$.

There are no intervals left, and therefore the final approximation is $\approx 0.38145 + 0.32338 + 0.28912 = 0.99395$ whereas the exact integral is $\int_0^{\pi/2} \cos(x) = 1$. Thus the error is $\approx |0.99395 - 1| \approx 0.006 < 0.05$.

(c) In this case, $f(x) = \exp(x)$, $a_0 = 0$, $b_0 = 1$.

$S[0, 1] = (1 + e)/2 \approx 1.85914$, $S[0, 0.5] = (1 + \exp(0.5))/4 \approx 0.66218$, $S[0.5, 1] = (\exp(0.5) + \exp(1))/4 \approx 1.09175$. $|S[a, b] - S[a, c] - S[c, b]| \approx 0.1052 < 3 \cdot 0.05 \cdot ((b - a)/(b_0 - a_0)) = 0.15$. Thus we stop with the approximation $S[a, c] + S[c, b] \approx 1.75393$ and an error estimate $-(S[a, b] - S[a, c] - S[c, b])/3 \approx -0.035$.

The actual integration error is $\int_0^1 \exp(x) - (S[a, c] + S[c, b]) \approx -0.036$. Our estimate is quite good in this case.

7 Exercise 5.4.4

Develop an Adaptive Quadrature method for rule (5.28).

Solution

Let $c = (a + b)/2$, $h = b - a$ and apply the quadrature on $[a, b]$, $[a, c]$, and $[c, b]$ (we use the same notation as in the book):

$$\int_a^b f(x)\,\mathrm{d}x = S[a, b] + \frac{14h^5}{45} f^{(4)}(c_1), \qquad c_1 \in [a, b],$$

$$\int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x = S[a, c] + S[c, b] + \frac{14(h/2)^5}{45} f^{(4)}(c_2) + \frac{14(h/2)^5}{45} f^{(4)}(c_3), \qquad c_2 \in [a, c], c3 \in [c, b].$$

Further assuming that $f^{(4)}(c_1) \approx f^{(4)}(c_2) \approx f^{(4)}(c_3)$ we obtain:

$$\int_a^b f(x)\,\mathrm{d}x = S[a, b] + \frac{14h^5}{45} f^{(4)}(c_1),$$

$$\int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x \approx S[a, c] + S[c, b] + \frac{1}{16}\frac{14h^5}{45} f^{(4)}(c_1).$$

We can now subtract the second equation from the first to obtain:

$$S[a, b] - S[a, c] - S[c, b] \approx 15 \frac{1}{16} \frac{14h^5}{45} f^{(4)}(c_1).$$

Thus the number $S[a,b] - S[a,c] - S[c,b]$ gives an approximation to 15 times the error of the quadrature $S[a,c] + S[c,b]$.

The rest is exactly the same as for other quadratures; one stops subdividing the interval when $|S[a,b] - S[a,c] - S[c,b]| < 15 \cdot TOL \cdot (b-a)/(b_{\text{orig}} - a_{\text{orig}})$ and returns $S[a,c] + S[c,b]$ as the approximation of the integral over the interval [a,b].

8 Computer exercise 5.2.8

Answers:

(a) 1.547866

(b) 1.277978

(c) 1.277978

9 Computer exercise 5.2.10

See the file uniform_refinement.py available on the wiki. Adapt the code to use Simpson's rule instead of Trapezoid.