[S]=T. Sauer, Numerical Analysis, Second International Edition, Pearson, 2014

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1 Oppgave 6.4.4, (a), (b) s. 321, [S]

Solution:

Use formulae (6.50) on p.316 [S].

(a):

```
t_i=0.000000e+00 w_i=0.000000e+00
t_i=0.000000e+00 w_i=0.000000e+00 s1=0.000000e+00 s2=1.250000e-01 s3=1.406250e-01 s4=2.851562e-01 w_{i+1}=3.401693e-02
t_i=2.500000e-01 w_i=3.401693e-02 s1=2.840169e-01 s2=4.445190e-01 s3=4.645818e-01 s4=6.501624e-01 w_{i+1}=1.486995e-01
t_i=5.000000e-01 w_i=1.486995e-01 s1=6.486995e-01 s2=8.547869e-01 s3=8.805478e-01 s4=1.118836e+00 w_{i+1}=3.669580e-01
t_i=7.500000e-01 w_i=3.669580e-01 s1=1.116958e+00 s2=1.381578e+00 s3=1.414655e+00 s4=1.720622e+00 w_{i+1}=7.182099e-01
t_i=1.000000e+00 w_i=7.182099e-01
error(t=1)=7.188926e-05
```

(b):

```
t_i=0.000000e+00 w_i=0.000000e+00
t_i=0.000000e+00 w_i=0.000000e+00 s1=0.000000e+00 s2=1.250000e-01 s3=1.093750e-01 s4=2.226562e-01 w_{i+1}=2.880859e-02
t_i=2.500000e-01 w_i=2.880859e-02 s1=2.211914e-01 s2=3.185425e-01 s3=3.063736e-01 s4=3.945980e-01 w_{i+1}=1.065428e-01
t_i=5.000000e-01 w_i=1.065428e-01 s1=3.934572e-01 s2=4.692750e-01 s3=4.597978e-01 s4=5.285077e-01 w_{i+1}=2.223808e-01
t_i=7.500000e-01 w_i=2.223808e-01 s1=5.276192e-01 s2=5.866668e-01 s3=5.792859e-01 s4=6.327978e-01 w_{i+1}=3.678942e-01
t_i=1.000000e+00 w_i=3.678942e-01
error(t=1)=1.475824e-05
```

2 Oppgave 6.4.5, s. 321, [S]

Solution: To compute the one-step error we substitute $w_i = y(t_i)$ into (6.49) and perform a Taylor series expansion:

$$\begin{aligned}
w_{i+1} &= y(t_i) + (h - \tfrac{1}{2\alpha}) f(t_i, y(t_i)) + \tfrac{h}{2\alpha} f(t_i + \alpha h, y(t_i) + \alpha h f(t_i, y(i))) \\
&= y(t_i) + (h - \tfrac{1}{2\alpha}) \underbrace{f(t_i, y(t_i))}_{=y'(t_i)} + \tfrac{h}{2\alpha} \left[\underbrace{f(t_i, y(t_i))}_{=y'(t_i)} + \frac{\partial f}{\partial t}(t_i, y(t_i))\alpha h + \frac{\partial f}{\partial y}(t_i, y(t_i))\alpha h \underbrace{f(t_i, y(i))}_{=y'(t_i)} + O(h^2) \right] \\
&= y(t_i) + h y'(t_i) + \tfrac{h^2}{2} \underbrace{\left[\frac{\partial f}{\partial t}(t_i, y(t_i)) + \frac{\partial f}{\partial y}(t_i, y(t_i)) y'(t_i) \right]}_{=y''(t_i)} + O(h^3),
\end{aligned}$$

because $y''(t_i) = [f(t, y(t))]'_{t=t_i} = \frac{\partial f}{\partial t}(t_i, y(t_i)) + \frac{\partial f}{\partial y}(t_i, y(t_i)) y'(t_i)$ owing to the chain rule.

Thus w_{i+1} agrees with the Taylor series expantion for $y(t_{i+1}) = y(t_i + h)$ up to the second order terms, and the one step error is therefore $O(h^3)$. The global error is then $O(h^2)$.

$\boxed{3}$ Oppgave 6.4.6, s. 321, [S]

Solution:

$$
\begin{aligned}
&w_0 = y_0\\
&s_1 = f(t_0, w_0) = \lambda w_0\\
&s_2 = f(t_0 + h/2, w_0 + h/2 s_1) = \lambda(w_0 + h/2\lambda w_0 = w_0(\lambda + h/2\lambda^2)\\
&s_3 = f(t_0 + h/2, w_0 + h/2 s_3) = w_0(\lambda + h/2\lambda^2 + h^2/4\lambda^3)\\
&s_4 = f(t_0 + h, w_0 + h s_3) = w_0(\lambda + h\lambda^2 + h^2/2\lambda^3 + h^3/4\lambda^4)\\
&w_1 = w_0 + h/6(s_1 + 2s_2 + 2s_3 + s_4)\\
&\quad = w_0 + \frac{w_0 h}{6}[(1+2+2+1)\lambda + (0 + 2\tfrac{h}{2} + 2\tfrac{h}{2} + h)\lambda^2 + (0 + 0 + 2\tfrac{h^2}{4} + \tfrac{h^2}{2})\lambda^3 + (0 + 0 + 0 + \tfrac{h^3}{4})\lambda^4]\\
&\quad = w_0[1 + \lambda h + \lambda^2 \tfrac{h^2}{2} + \lambda^3 \tfrac{h^3}{6} + \lambda^4 \tfrac{h^4}{24}].
\end{aligned}
$$

Thus w_1 produced by RK4 agrees with a Taylor series expansion of $y(h) = y_0 \exp(\lambda h)$ all the way up to terms of order h^4. Thus one-step error is $O(h^5)$.

$\boxed{4}$ Oppgave 6.4.7, s. 321, [S]

Solution:

$$
\begin{aligned}
w_i &= f(t_i)\\
s_1 &= f(t_i, w_i) = f(t_i)\\
s_2 &= f(t_i + h/2, w_i + h/2 s_1) = f(t_i + h/2)\\
s_3 &= f(t_i + h/2, w_i + h/2 s_3) = f(t_i + h/2)\\
s_4 &= f(t_i + h, w_0 + h s_3) = f(t_i + h)\\
w_1 &= w_0 + h/6(s_1 + 2s_2 + 2s_3 + s_4)\\
&= w_0 + \tfrac{h}{6}[f(t_i) + 4f(t_i + h/2) + f(t_i + h)],
\end{aligned}
$$

which is exactly the Simpsons rule for $\int_{t_i}^{t_i+h} f(t)\,\mathrm{d}t$.

$\boxed{5}$ Oppgave 6.6.1 (a), (b), s. 335, [S]

Solution:

Use formulae (6.69) on p.316 [S].

(a): In this case

$$
w_{i+1} = w_i + hf(t_i + h, w_{i+1}) = w_i + h(t_i + h + w_{i+1}) \qquad \text{and therefore}
$$

$$
w_{i+1} = \frac{w_i + ht_i + h^2}{1 - h}
$$

```
t_i=0.000000e+00 w_i=0.000000e+00
t_i=0.000000e+00 w_i=0.000000e+00 w_{i+1}=8.333333e-02
t_i=2.500000e-01 w_i=8.333333e-02 w_{i+1}=2.777778e-01
t_i=5.000000e-01 w_i=2.777778e-01 w_{i+1}=6.203704e-01
t_i=7.500000e-01 w_i=6.203704e-01 w_{i+1}=1.160494e+00
t_i=1.000000e+00 w_i=1.160494e+00
error(t=1)=4.422120e-01
```

(b): In this case

$$w_{i+1} = w_i + hf(t_i + h, w_{i+1}) = w_i + h(t_i + h - w_{i+1}) \qquad \text{and therefore}$$
$$w_{i+1} = \frac{w_i + ht_i + h^2}{1+h}$$

```
t_i=0.000000e+00 w_i=0.000000e+00
t_i=0.000000e+00 w_i=0.000000e+00 w_{i+1}=5.000000e-02
t_i=2.500000e-01 w_i=5.000000e-02 w_{i+1}=1.400000e-01
t_i=5.000000e-01 w_i=1.400000e-01 w_{i+1}=2.620000e-01
t_i=7.500000e-01 w_i=2.620000e-01 w_{i+1}=4.096000e-01
t_i=1.000000e+00 w_i=4.096000e-01
error(t=1)=4.172056e-02
```

$\boxed{6}$ Oppgave 6.6.4, s. 335, [S]

Solution:

(a): Let us find the solution to the ODE first. A particular solution can be sought in the form $y(t) = \alpha t + \beta$. After substituting this into the ODE we get that $\alpha = 0$ and $\beta = -b/a$. The general solution is then $-b/a + C\exp(at)$, where C is a constant to be derermined from the initial conditions.

Since $a < 0$ the solution converges to the equilibrium $-b/a$ from any starting point, as $t \to \infty$.

(b):

$$w_{i+1} = w_i + hf(t_i + h, w_{i+1}) = w_i + h(aw_{i+1} + b) \qquad \text{and therefore}$$
$$w_{i+1} = \frac{w_i + hb}{1 - ah} =: T(w_i)$$

We can view this method as a fixed-point iteration. Its only fixed point is computed from $w = T(w)$, or $w(1 - ah) = w + bh$, thus $w = -b/a$. This fixed point iteration is linear and will converge from any point when $|dT/dw| < 1$. This is the case because $|dT/dw| = |1/(1 - ah)| < 1$ because $a < 0$ and $h > 0$.

$\boxed{7}$ Consider the initial value problem

$$\begin{aligned} y' &= \lambda y, \qquad t > 0, \\ y(0) &= y_0, \end{aligned}$$

where $\lambda \in \mathbb{C}$. Its solution is $y(t) = y_0 \exp(\lambda t)$.

a) Suppose that we use a numerical method (such as e.g. forward Euler or explicit trapezoid) to solve this problem starting from a point $w_0 = y_0$. *The stability region* for the method is a set of points $z = \lambda h$ in the complex plane, such that the numerical solution $(w_0, w_1, \dots)$ stays *bounded* (i.e., $\exists C > 0 : \forall i, |w_i| \leq C$). Find the stability region for (1) implicit (backward) Euler method, defined by the formula $w_{i+1} = w_i + hf(t_{i+1}, w_{i+1})$, see p. 333 in [S]; (2) implicit Trapezoid method defined by the formula $w_{i+1} = w_i + h/2[f(t_i, w_i) + f(t_{i+1}, w_{i+1})]$.

Solution:
For the implicit Euler we get: $w_0 = y_0$,

$$w_{i+1} = w_i + hf(t_i + h, w_{i+1}) = w_i + \lambda h w_{i+1}, \qquad \text{and thus}$$
$$w_{i+1} = \frac{1}{1-\lambda h} w_i = \frac{1}{(1-\lambda h)^{k+1}} w_0.$$

The solution is bounded only when $|1 - \lambda h|^{-1} \leq 1$, that is, when $|\lambda h - 1| \geq 1$. Thus λh must be outside of the unit circle in the complex plain centered around 1.

For the implicit trapezoid we obtain

$$w_{i+1} = w_i + h/2[f(t_i, w_i) + f(t_i + h, w_{i+1})] = w_i + \lambda h/2[w_i + w_{i+1}], \qquad \text{and thus}$$
$$w_{i+1} = \frac{1+\lambda h/2}{1-\lambda h/2} w_i = \left(\frac{1+\lambda h/2}{1-\lambda h/2}\right)^{k+1} w_0.$$

The solution is bounded only when $|(1+\lambda h/2)/(1-\lambda h/2)| \leq 1$. Let us study this further (we use the notation $z = \lambda h$, and $\bar{z}$ to denote the complex conjugate):

$$\left|\frac{1+z/2}{1-z/2}\right| = \left(\frac{1+z/2}{1-z/2}\frac{1+\bar{z}/2}{1-\bar{z}/2}\right)^{1/2} = \left(\frac{1+z/2+\bar{z}/2+z\bar{z}/4}{1-z/2-\bar{z}/2+z\bar{z}/4}\right)^{1/2} = \left(\frac{1+\mathrm{Re}(z)+|z|^2/4}{1-\mathrm{Re}(z)+|z|^2/4}\right)^{1/2}$$

Thus the solution remains bounded only when the numerator is no larger in magnitude than the denominator, or only when $\mathrm{Re}(z) \leq 0$. Thus the stability region is the whole left half-plane.

b) Let $\lambda = j\omega$, where $j^2 = -1$ and $\omega > 0$. Show that the implicit trapezoid method matches the *amplitude* of the solution exactly, that is, $|w_i| = |y(t_i)|$, for all $i = 1, 2, \dots$

Solution:
Using the previous computations we have

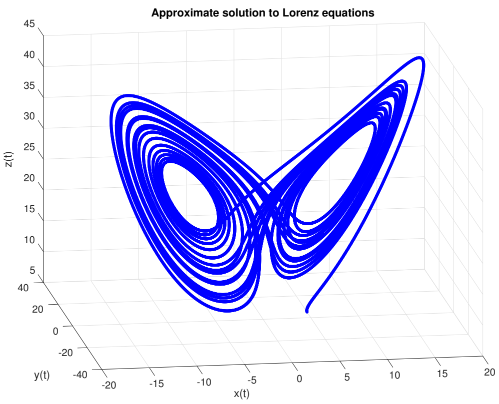
$$|w_k| = \left(\frac{1+\mathrm{Re}(z)+|z|^2/4}{1-\mathrm{Re}(z)+|z|^2/4}\right)^{k/2}\Bigg|_{z=j\omega h} |w_0| = |y_0|,$$

because $\mathrm{Re}(j\omega h) = 0$ and $w_0 = y_0$. The same holds for the exact solution: $|y(t)| = |\exp(j\omega t)y_0| = |y_0|$.

"Computeroppgaver"

$\boxed{8}$ Oppgave 6.4.12, s. 322, [S]

Solution:



Starting from $(x, y, z) = (5, 5, 5)$ the end point at $t = 10$ is approximately $(x, y, z) =$ (`2.113909e+00,3.723605e+00,1.139524e+01`), and at $t = 20$ is approximately $(x, y, z) =$ `-9.982849e+00,-1.602445e+01,1.929328e+01`).

Starting from $(x, y, z) = (5 + 10^{-5}, 5, 5)$ the end point at $t = 10$ is $(x, y, z) =$ (`2.114883e+00,3.725323e+00,1.139553e+01`), and at $t = 20$ is approximately $(x, y, z) =$ `1.124423e+01,1.881841e+01,1.892956e+01`).

The error amplification is distance at the end divided by the distance at the beginning, or `2E-03/1E-05` ≈ 200 at $t = 10$, and approximately `4E+06` at $t = 20$.

$\boxed{9}$ Oppgave 6.6.2, s. 336, [S].

Solution: see `oppgave_6_6_2.m`

(a): Equilibrium point: $y \equiv 2$. Explicit Euler converges to this equilibrium for $h \lessapprox 0.3$.

(b): Equilibrium point: $y \equiv 1$. Explicit Euler converges for $h \lessapprox 0.2$.