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Here is a collection of some exam questions taken from the prevous exams in TMA4215 and MA2501 (sometimes modified), which are pertinent to TMA4320.

≈Problem 1, Exam MA2501, 10.06.2010 Gitt tatasettet:

x	1	1.5	2.0
y	-1	3	3

a) Finn polynomet $p(x)$ av lavest mulig grad som interpolerer datasettet.

b) Bestem konstantene a, b, og c slik at $p(x)$ interpolerer funksjonen

$$f(x) = a\cos(\pi x) + b\sin(\pi x) + c$$

i de tre punktene $x = 1$, $x = 1.5$, og $x = 2$.

c) Finn en øvre grense for feilen $|f(x) - p(x)|$ for $1 \le x \le 2$. Husk at

$$f(x) = p_n(x) + \frac{f^{(n)}(\xi)}{n!}\prod_{i=1}^{n}(x - x_i), \quad x, \xi \in [\min_i x_i, \max_i x_i].$$

≈Problem 1, Exam MA2501, 25.05.2011 Consider the function $f(x) = |x|$. We want to interpolate the function at the points -2, -1.5, 0.5, 1.

a) Express the interpolation polynomial using the Lagrange polynomials.

b) Use Newton's divided difference method to find the interpolation polynomial.

≈Problem 2, Exam MA2501, 25.05.2011 Consider the differential equation $u''(t) = f(u(t))$ for an arbitrary function $f : \mathbb{R} \to \mathbb{R}$.

a) Convert the differential equation into a system of first order equations and write down one step of the explicit Euler method.

≈Problem 3, Exam MA2501, 25.05.2011 We want to solve the following equation in u_1 (u_1 is the unknown, whereas u_0 and h are known):

$$u_1 = u_0 + h\cos(u_1)$$

a) Run one iteration of Newton's method to find an approximation of u_1.

b) Assume for instance that $u_0 = 1$ and $h = 0.1$. Choose an initial guess for Newton's method and compute the result after one step of Newton's method using the formula in part **a)**.

≈Problem 5, Exam MA2501, 25.05.2011 On the interval $[0, 1]$ we choose the interpolation points

$$x_1 = \frac{1}{4}, \quad x_2 = \frac{1}{2}, \quad x_3 = \frac{3}{4}.$$

One obtains a local integration formula, i.e., an approximation of $\int_0^1 f(x)\,\mathrm{d}x$, by interpolating f at the interpolation points, and by integrating exactly the resulting interpolation polynomial. The resulting formula takes the form

$$I(f) = \sum_{i=1}^{3} w_i f(x_i).$$

a) Find out the weights w_i for the formula above.

b) Show that this formula integrates exactly polynomials of degree lower than or equal to 3, but not exactly all polynomials of degree 4.

≈Problem 2, Exam MA2501, 07.06.2014 Consider the function

$$f(x) = 2x - \sin(x) + 2.$$

In order to solve the equation $f(x) = 0$, it is possible to apply a fixed point iteration of the form

$$x_{k+1} = x_k - \frac{1}{2} f(x_k).$$

a) Show that the equation $f(x) = 0$ has a unnique solution $\hat{x}$, and that the iteration converges for every starting value $x_0 \in \mathbb{R}$ to $\hat{x}$.

b) Compute one step of the fixed point iteration with a starting value $x_0 = 0$. Use your result to estimate, after how many steps we have $|x_k - \hat{x}| \leq 2^{-20}$.

≈Problem 3, Exam MA2501, 07.06.2014 Denote by f_n the polynomial of degree n that interpolates the function $f(x) = e^x + e^{-x}$ in equidistant interpolation points on the interval $[0, 1]$.

a) Provide an estimate for

$$\sup_{0 \leq x \leq 1} |f_5(x) - f(x)|.$$

Hint: use the representation

$$f(x) = f_n(x) + \frac{f^{(n)}(\xi)}{n!} \prod_{i=1}^{n} (x - x_i), \quad \xi \in [\min\{x_i, x\}, \max\{x_i, x\}].$$

b) Show that $\lim_{n \to \infty} f_n(x) = f(x)$ for every $x \in \mathbb{R}$. Hint: use the same representation as in **a)** and Stirling's estimate of $n!$:

$$n! \geq \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

≈Problem 5, Exam MA2501, 07.06.2014 Consider a quadrature rule of the form

$$Q(f, -1, 1) = 2[c_0 f(-1) + c_1 f(-2/3) + c_2 f(0) + c_3 f(2/3) + c_4 f(1)]$$

for the approximation of a definite integral $\int_{-1}^{1} f(x)\,\mathrm{d}x$.

a) Find weights $c_0, \ldots, c_4 \in \mathbb{R}$ such that all polynomials of degree 4 are integrated exactly.

b) Using the weights computed in the first part of the exercise, find the smallest integer $k \in \mathbb{N}$ such that $Q(x^k, -1, 1) \neq \int_{-1}^{1} x^k\,\mathrm{d}x$.

≈Problem 6, Exam MA2501, 07.06.2014 Consider the initial value problem

$$y' = \cos(y) - 2y, \quad y(0) = 0.$$

a) Apply two steps of the explicit Euler method with a step size of $h = 1$ for the solution of this equation.

b) Apply two steps of the implicit Euler method with a step size of $h = 1$ for the solution of this equation. In each step, use two steps of Newton's method (with a reasonable starting value of your choice) for the solution of the non-linear equation you have to solve.

≈Problem 1, Exam MA2501, 04.06.2015 Approximate the value of the integral

$$\int_0^1 e^{-x}\,\mathrm{d}x$$

using the composite Simpson's rule with $n = 4$ panels. Determine the upper bound for the absolute error. Verify that the absolute error is within this bound.

Hint: A *non-composite* Simpson's rule is given by

$$\int_a^b f(x)\,\mathrm{d}x = \frac{1}{3}\frac{b-a}{2}[f(a) + 4f((a+b)/2) + f(b)] - \frac{1}{90}\left(\frac{b-a}{2}\right)^5 |f^{(4)}(\xi)|,$$

where $\xi \in [a, b]$.

≈Problem 4, Exam MA2501, 04.06.2015 **a)** The function $f(x) = \sin x$ has the unique zero $x^* = \pi$ on the interval $[3, 4]$ Perform 3 iterations of Newton's method to approximate this zero. Use $x_0 = 4$.

Note: do not round off the calculated values, they will be used in what follows.

b) Determine the absolute error for x_i, $i = 0, 1, 2, 3$ from **a)**. Estimate the order of convergence α.

Hint: you can assume that $\alpha \in \mathbb{N}$. A method with order of convergence $\alpha \in \mathbb{N}$ will behave like

$$|x_{n+1} - x^*| \approx M|x_n - x^*|^\alpha,$$

for some positive constant M when $|x_n - x^*|$ becomes sufficiently small.

c) Explain the order of convergence observed in **b)**.

≈Problem 6, Exam MA2501, 04.06.2015 Consider the following initial value problem for $x : \mathbb{R} \to \mathbb{R}$ and $y : \mathbb{R} \to \mathbb{R}$:

$$\begin{aligned} x''(t) &= e^{-x'(t)} + x(t) - \cos(t), \\ y'(t) &= [y(t)]^{1/3} - tx'(t), \\ x(0) &= -2, \quad x'(0) = 0, \quad y(0) = 8. \end{aligned}$$

Convert this problem to an equivalent initial value problem for a system of first-order differential equations. Take 2 steps with Euler's method with step-size $h = 0.5$ for the resulting system.

≈Problem 1, Exam MA2501, 31.05.2013 **a)** Find the polynomial $p(x)$ of lowest degree that interpolates

$$f(x) = (x-2)^{1/3}$$

at the points 3, 3.7, 4.

b) Determine an upper bound for the absolute error $|f(x) - p(x)|$ on the interval $[3, 4]$.
Hint:

$$f(x) = p_n(x) + \frac{f^{(n)}(\xi)}{n!}\prod_{i=1}^{n}(x - x_i), \quad x, \xi \in [\min_i x_i, \max_i x_i].$$

≈Problem 3, Exam MA2501, 31.05.2013 Let $f(x) = 5/sqrtx$ and consider the fixed-point iteration $x_{n+1} = f(x_n)$, $n \geq 0$.

a) Determine the fixed points of this iteration.

b) Will the iteration converge if we start "close enough" to the fixed points?

c) Perform 4 steps of this method starting from $x_0 = 3$.

Consider a related fixed point iteration defined by $x_{n+1} = g(x_n)$ where

$$g(x) = \frac{2x^3 + 25}{3x^2}.$$

d) Determine the fixed points of this iteration.

e) Perform 4 steps of the fixed point iteration defined by g starting from $x_0 = 3$.

f) Determine the convergence rate of both fixed-point iteration processes.
Hint: A method with order of convergence $\alpha \in \mathbb{N}$ will behave like

$$|x_{n+1} - x^*| \approx M|x_n - x^*|^{\alpha},$$

for some positive constant M when $|x_n - x^*|$ becomes sufficiently small.

≈Problem 1, Exam MA2501, 08.06.2012 Consider the interpolation points

x	1	2	3	4
y	1	5	15	7

a) Compute the Lagrange polynomial for the interpolation point $x = 4$.

b) Use divided differences to compute the interpolating polynomial of minimum degree.

≈Problem 2, Exam MA2501, 08.06.2012 Consider the differential equation

$$u'(t) = -u(t) + t + \frac{1}{2}$$

with the initial condition $u(0) = 1$. Use the explicit Euler method in order to compute an approximation of $u(0.2)$, using a time step $h = 0.1$.

≈Problem 3, Exam MA2501, 08.06.2012 Consider the nodes

$$c_1 = \frac{1}{6}, \quad c_2 = \frac{1}{2}, \quad c_3 = \frac{5}{6},$$

and the corresponding quadrature formula

$$Q(f) = w_1 f(c_1) + w_2 f(c_2) + w_3 f(c_3)$$

for approximating the integral $\int_0^1 f(x)\,\mathrm{d}x$.

a) Determine the weights w_1, w_2, w_3 such that the quadrature is exact for polynomials of degree up to two.

b) Compute the error $E_k = |Q(x^k) - \int_0^1 x^k\,\mathrm{d}x|$ for the lowest integer k such that E_k is not zero.