Here is a collection of some exam questions taken from the previous exams in TMA4215 and MA2501 (sometimes modified), which are pertinent to TMA4320.

≈Problem 1, Exam MA2501, 10.06.2010≈

Gitt datassetet:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>1.5</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>y</td>
<td>-1</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

a) Finn polynomet $p(x)$ av lavest mulig grad som interpolerer datassetet.

b) Bestem konstantene $a$, $b$, og $c$ slik at $p(x)$ interpolerer funksjonen

$$f(x) = a \cos(\pi x) + b \sin(\pi x) + c$$

i de tre punktene $x = 1$, $x = 1.5$, og $x = 2$.

c) Finn en øvre grense for feilen $|f(x) - p(x)|$ for $1 \leq x \leq 2$. Husk at

$$f(x) = p_n(x) + \frac{f^{(n)}(\xi)}{n!} \prod_{i=1}^{n}(x - x_i), \quad x, \xi \in [\min x_i, \max x_i].$$
Problem 1, Exam MA2501, 25.05.2011
Consider the function \( f(x) = |x| \). We want to interpolate the function at the points \(-2, -1.5, 0.5, 1\).

a) Express the interpolation polynomial using the Lagrange polynomials.

b) Use Newton’s divided difference method to find the interpolation polynomial.

Problem 2, Exam MA2501, 25.05.2011
Consider the differential equation \( u''(t) = f(u(t)) \) for an arbitrary function \( f : \mathbb{R} \to \mathbb{R} \).

a) Convert the differential equation into a system of first order equations and write down one step of the explicit Euler method.

Problem 3, Exam MA2501, 25.05.2011
We want to solve the following equation in \( u_1 \) (\( u_1 \) is the unknown, whereas \( u_0 \) and \( h \) are known):

\[
    u_1 = u_0 + h \cos(u_1)
\]

a) Run one iteration of Newton’s method to find an approximation of \( u_1 \).

b) Assume for instance that \( u_0 = 1 \) and \( h = 0.1 \). Choose an initial guess for Newton’s method and compute the result after one step of Newton’s method using the formula in part a).

Problem 5, Exam MA2501, 25.05.2011
On the interval \([0, 1]\) we choose the interpolation points

\[
    x_1 = \frac{1}{4}, \quad x_2 = \frac{1}{2}, \quad x_3 = \frac{3}{4}.
\]

One obtains a local integration formula, i.e., an approximation of \( \int_0^1 f(x) \, dx \), by interpolating \( f \) at the interpolation points, and by integrating exactly the resulting interpolation polynomial. The resulting formula takes the form

\[
    I(f) = \sum_{i=1}^{3} w_i f(x_i).
\]

a) Find out the weights \( w_i \) for the formula above.

b) Show that this formula integrates exactly polynomials of degree lower than or equal to 3, but not exactly all polynomials of degree 4.
Problem 2, Exam MA2501, 07.06.2014

Consider the function
\[ f(x) = 2x - \sin(x) + 2. \]

In order to solve the equation \( f(x) = 0 \), it is possible to apply a fixed point iteration of the form
\[ x_{k+1} = x_k - \frac{1}{2} f(x_k). \]

a) Show that the equation \( f(x) = 0 \) has a unique solution \( \hat{x} \), and that the iteration converges for every starting value \( x_0 \in \mathbb{R} \) to \( \hat{x} \).

b) Compute one step of the fixed point iteration with a starting value \( x_0 = 0 \). Use your result to estimate, after how many steps we have \( |x_k - \hat{x}| \leq 2^{-20} \).

Problem 3, Exam MA2501, 07.06.2014

Denote by \( f_n \) the polynomial of degree \( n \) that interpolates the function \( f(x) = e^x + e^{-x} \) in equidistant interpolation points on the interval \([0, 1]\).

a) Provide an estimate for \( \sup_{0 \leq x \leq 1} |f_5(x) - f(x)| \).

Hint: use the representation
\[ f(x) = f_n(x) + \frac{f^{(n)}(\xi)}{n!} \prod_{i=1}^{n} (x - x_i), \quad \xi \in [\min\{x_i, x\}, \max\{x_i, x\}]. \]

b) Show that \( \lim_{n \to \infty} f_n(x) = f(x) \) for every \( x \in \mathbb{R} \). Hint: use the same representation as in a) and Stirling’s estimate of \( n! \):
\[ n! \geq \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \]

Problem 5, Exam MA2501, 07.06.2014

Consider a quadrature rule of the form
\[ Q(f, -1, 1) = 2[c_0 f(-1) + c_1 f(-2/3) + c_2 f(0) + c_3 f(2/3) + c_4 f(1)] \]

for the approximation of a definite integral \( \int_{-1}^{1} f(x) \, dx \).

a) Find weights \( c_0, \ldots, c_4 \in \mathbb{R} \) such that all polynomials of degree 4 are integrated exactly.

b) Using the weights computed in the first part of the exercise, find the smallest integer \( k \in \mathbb{N} \) such that \( Q(x^k, -1, 1) \neq \int_{-1}^{1} x^k \, dx \).

Problem 6, Exam MA2501, 07.06.2014

Consider the initial value problem
\[ y' = \cos(y) - 2y, \quad y(0) = 0. \]
a) Apply two steps of the explicit Euler method with a step size of $h = 1$ for the solution of this equation.

b) Apply two steps of the implicit Euler method with a step size of $h = 1$ for the solution of this equation. In each step, use two steps of Newton’s method (with a reasonable starting value of your choice) for the solution of the non-linear equation you have to solve.
Problem 1, Exam MA2501, 04.06.2015
Approximate the value of the integral
\[ \int_{0}^{1} e^{-x} \, dx \]
using the composite Simpson’s rule with \( n = 4 \) panels. Determine the upper bound for the absolute error. Verify that the absolute error is within this bound.

Hint: A non-composite Simpson’s rule is given by
\[ \int_{a}^{b} f(x) \, dx = \frac{1}{3} \frac{b-a}{2} [f(a) + 4f((a+b)/2) + f(b)] - \frac{1}{150} \left( \frac{b-a}{2} \right)^5 |f^{(4)}(\xi)|, \]
where \( \xi \in [a, b] \).

Problem 4, Exam MA2501, 04.06.2015
a) The function \( f(x) = \sin x \) has the unique zero \( x^* = \pi \) on the interval \([3, 4]\) Perform 3 iterations of Newton’s method to approximate this zero. Use \( x_0 = 4 \).

Note: do not round off the calculated values, they will be used in what follows.

b) Determine the absolute error for \( x_i, i = 0, 1, 2, 3 \) from a). Estimate the order of convergence \( \alpha \).

Hint: you can assume that \( \alpha \in \mathbb{N} \). A method with order of convergence \( \alpha \in \mathbb{N} \) will behave like
\[ |x_{n+1} - x^*| \approx M |x_n - x^*|^\alpha, \]
for some positive constant \( M \) when \( |x_n - x^*| \) becomes sufficiently small.

c) Explain the order of convergence observed in b).

Problem 6, Exam MA2501, 04.06.2015
Consider the following initial value problem for \( x : \mathbb{R} \to \mathbb{R} \) and \( y : \mathbb{R} \to \mathbb{R} \):
\[ x''(t) = e^{-x'(t)} + x(t) - \cos(t), \]
\[ y'(t) = |y(t)|^{1/3} - tx'(t), \]
\[ x(0) = -2, \quad x'(0) = 0, \quad y(0) = 8. \]

Convert this problem to an equivalent initial value problem for a system of first-order differential equations. Take 2 steps with Euler’s method with step-size \( h = 0.5 \) for the resulting system.
Problem 1, Exam MA2501, 31.05.2013

**a)** Find the polynomial \( p(x) \) of lowest degree that interpolates
\[
f(x) = (x - 2)^{1/3}
\]
at the points 3, 3.7, 4.

**b)** Determine an upper bound for the absolute error \( |f(x) - p(x)| \) on the interval [3, 4].

Hint:
\[
f(x) = p_n(x) + \frac{f^{(n)}(\xi)}{n!} \prod_{i=1}^{n} (x - x_i), \quad x, \xi \in [\min_i x_i, \max_i x_i].
\]

Problem 3, Exam MA2501, 31.05.2013

Let \( f(x) = \frac{5}{\sqrt{x}} \) and consider the fixed-point iteration \( x_{n+1} = f(x_n), \ n \geq 0. \)

**a)** Determine the fixed points of this iteration.

**b)** Will the iteration converge if we start “close enough” to the fixed points?

**c)** Perform 4 steps of this method starting from \( x_0 = 3. \)

Consider a related fixed point iteration defined by \( x_{n+1} = g(x_n) \) where
\[
g(x) = \frac{2x^3 + 25}{3x^2}.
\]

**d)** Determine the fixed points of this iteration.

**e)** Perform 4 steps of the fixed point iteration defined by \( g \) starting from \( x_0 = 3. \)

**f)** Determine the convergence rate of both fixed-point iteration processes.

Hint: A method with order of convergence \( \alpha \in \mathbb{N} \) will behave like
\[
|x_{n+1} - x^*| \approx M|x_n - x^*|^\alpha,
\]
for some positive constant \( M \) when \( |x_n - x^*| \) becomes sufficiently small.
Problem 1, Exam MA2501, 08.06.2012
Consider the interpolation points
\[
\begin{array}{c|cccc}
  x & 1 & 2 & 3 & 4 \\
  y & 1 & 5 & 15 & 7 \\
\end{array}
\]
a) Compute the Lagrange polynomial for the interpolation point \( x = 4 \).
b) Use divided differences to compute the interpolating polynomial of minimum degree.

Problem 2, Exam MA2501, 08.06.2012
Consider the differential equation
\[ u'(t) = -u(t) + t + \frac{1}{2} \]
with the initial condition \( u(0) = 1 \). Use the explicit Euler method in order to compute an approximation of \( u(0.2) \), using a time step \( h = 0.1 \).

Problem 3, Exam MA2501, 08.06.2012
Consider the nodes
\[ c_1 = \frac{1}{6}, \quad c_2 = \frac{1}{2}, \quad c_3 = \frac{5}{6}, \]
and the corresponding quadrature formula
\[ Q(f) = w_1 f(c_1) + w_2 f(c_2) + w_3 f(c_3) \]
for approximating the integral \( \int_0^1 f(x) \, dx \).
a) Determine the weights \( w_1, w_2, w_3 \) such that the quadrature is exact for polynomials of degree up to two.
b) Compute the error \( E_k = |Q(x^k) - \int_0^1 x^k \, dx| \) for the lowest integer \( k \) such that \( E_k \) is not zero.