[S]=T. Sauer, Numerical Analysis, Second International Edition, Pearson, 2014

1. Solution of equations Let $f(x) = ax + b$, where $a \neq 0$.

a) Let $x_1, x_2 \in \mathbb{R}$ be such that $f(x_1)f(x_2) < 0$. For which values of a, b, does the bisection algorithm applied to f converges starting from $[x_1, x_2]$.

Consider now a fixed point iteration $x_{k+1} = (a+1)x_k + b$.

b) Show that any fixed point of this iteration is a root of the equation $f(x) = 0$, and vice versa.

c) For which values a, b does the fixed point iteration algorithm converge for any starting point?

d) For which values a, b does the fixed point iteration algorithm exhibits *faster* convergence rate than the bisection algorithm?

2. Interpolation of functions Throughout this question we put $f(x) = x^3 + x^2 + 1$.

a) Compute the lowest degree polynomial, which passes through the points $(0, f(0))$, $(1, f(1))$, $(2, f(2))$.

b) Compute the lowest degree polynomial, which passes through the points $(0, f(0))$, $(1, f(1))$, $(2, f(2))$, $(\pi, f(\pi))$.

c) Let $p_0(x) = f(x_1)$ be the zeroth degree polynomial passing through the point $(x_1, f(x_1))$. Consider the problem of finding the point $x_1 \in [0, 2]$ such that the quantity $\max_{x\in[0,2]} |f(x) - p_0(x)|$ is as small as possible. Graphically show that $x_1 \approx 1.5377\ldots$, the point that satisfies the equation $f(x_1) = 7$.

d) The Chebyshev node for $n = 1$ on the interval $[0, 2]$ is $\hat{x}_1 = 1$ (the middle of the interval). In view of the optimality of Chebyshev interpolation, explain why it is possible for x_1 in the previous question *not* to coincide with $\hat{x}_1$?

3. Numerical integration Midpoint quadrature rule is defined by/satisfies the equation

$$\int_{x_0}^{x_1} f(x)\,\mathrm{d}x = hf(w) + \frac{h^3}{24} f''(c), \tag{1}$$

where $c \in [x_0, x_1]$, $h = x_1 - x_0$, $w = (x_0 + x_1)/2$, assuming that f is twice continuously differentiable on $[a, b]$.

a) Provide an upper (pessimistic) estimate the error for the mid-point approximation of $\int_0^1 \exp(x^2)\,\mathrm{d}x$.

b) Provide an estimate of the error term in (1) for the integral in **a)** using the adaptive quadrature idea, that is, by applying a composite midpoint quadrature on $[x_0, x_1]$ with two panels.

4. Solution of ODEs Consider the initial value problem $y''(t) + [y'(t)]^2 = 1$, $y(0) = y'(0) = 1$.

a) Rewrite the problem as a system of first order ODEs.

b) Apply one step of the explicit trapezoid method with steplength $h = 0.1$.

5. DFT **a)** Compute DFT of a sequence $[0, 1, 2, 3]$.

b) If the input sequence x is *real* than its DFT y satisfies the properties: (i) $y_0 \in \mathbb{R}$, (ii) $y_{n-i} = \bar{y}_i$, $i = 1, \dots, n-1$.

Suppose now that we know that the "output" sequence y (which is still DFT of x) is real. What can we say about x?