Solutions_3

January 27, 2016

1 Exercise 5.2.3

Apply composite Simpson's rule with $m = 1, 2, 4$ panels to approximate the integrals:

$$\text{(a)} \quad \int_0^1 x^2 \,\mathrm{d}x = \frac{1}{3}, \quad \text{(b)} \quad \int_0^{\pi/2} \cos(x) \,\mathrm{d}x = 1, \quad \text{(c)} \quad \int_0^1 e^x \,\mathrm{d}x = e - 1,$$

and report the errors.

Solution

(a) $f(x) = x^2$.

For $m = 1$ we put $h = (1-0)/2 = 1/2$, $x_0 = 0$, $x_1 = 1/2$, $x_2 = 1$; $y_0 = f(x_0) = 0$, $y_1 = f(x_1) = 1/4$, $y_2 = f(x_2) = 1$. The approximation is then $I \approx h/3(y_0 + 4y_1 + y_2) = (0 + 4 * 1/4 + 1)/6 = 1/3$. The error is thus 0.

For $m = 2$ we have $h = (1-0)/4 = 1/4$, $(x_0, x_1, \ldots, x_4) = (0, 1/4, 1/2, 3/4, 1)$, $(y_0, y_1, \ldots, y_4) = (0, 1/16, 1/4, 9/16, 1)$. The approximation is then $I \approx h/3(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) = (0 + 4 \cdot 1/16 + 2 \cdot 1/4 + 4 \cdot 9/16 + 1)/12 = (1/4 + 2/4 + 9/4 + 4/4)/12 = 16/(3 \cdot 4 \cdot 4) = 1/3$. The error is thus 0.

One could continue with $m = 3$, but the error has to be zero. Indeed, Simpson's rule is based on quadratic interpolation polynomials, which means that x^2 will be represented exactly by the interpolating polynomial and the quadrature will be exact.

(b) $f(x) = \cos(x)$

For $m = 1$ we put $h = (\pi/2 - 0)/2 = \pi/4$, $x_0 = 0$, $x_1 = \pi/4$, $x_2 = \pi/2$; $y_0 = f(x_0) = 1$, $y_1 = f(x_1) = 1/\sqrt{2}$, $y_2 = f(x_2) = 0$. The approximation is then $I \approx h/3(y_0 + 4y_1 + y_2) = (1 + 4/\sqrt{2} + 0) \cdot \pi/12 \approx 1.00227$. The error is thus ≈ 0.00227.

For $m = 2$, $h = \pi/8$, $(x_0, \ldots, x_4) = (0, \pi/8, \pi/4, 3\pi/8, \pi/2)$, $(y_0, \ldots, y_4) \approx (1, 0.9238795, 1/\sqrt{2}, 0.38268343, 0)$. $I \approx h/3(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) \approx 1.000135$, giving the error ≈ 0.000135.

For $m = 3$, $h = \pi/6$, $I \approx 1.00002631$, errror ≈ 0.00002631.

(c) $f(x) = \exp(x)$

For $m = 1$ we put $h = (1-0)/2 = 1/2$, $x_0 = 0$, $x_1 = 1/2$, $x_2 = 1$; $y_0 = f(x_0) = 1$, $y_1 = f(x_1) = e^{1/2}$, $y_2 = f(x_2) = e$. The approximation is then $I \approx h/3(y_0 + 4y_1 + y_2) = (1 + 4\sqrt{e} + e)/6 \approx 1.71886115$. The error is thus $\approx 1.71886115 - e + 1 \approx 5.79 \cdot 10^{-4}$.

For $m = 2$, $h = 1/4$, $(x_0, \ldots, x_4) = (0, 1/4, 1/2, 3/4, 1)$, $\ldots I \approx h/3(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) \approx 1.71831884192175$, giving the error $\approx 3.70 \cdot 10^{-5}$.

For $m = 3$, $h = 1/6$, $I \approx 1.71828916992083$, errror $\approx 7.34 \cdot 10^{-6}$.

2 Exercise 5.2.11

Find the degree of precision of the following approximation for $\int_{-1}^{1} f(x) \,\mathrm{d}x$:

First of all, let us evaluate the integral of a monomial x^n, $n \geq 0$:

$$\int_{-1}^{1} x^n \,\mathrm{d}x = \left[\frac{x^{n+1}}{n+1}\right]_{x=-1}^{x=1} = \begin{cases} 0, & \text{for odd } n, \\ 2/(n+1), & \text{for even } n. \end{cases}$$

(a) $f(1) + f(-1)$:
Let us apply the quadrature to the monomial x^n, $n \geq 0$:

$$1^n + (-1)^n = \begin{cases} 0, & \text{for odd } n, \\ 2, & \text{for even } n, \end{cases}$$

Therefore the quadrature agrees with the integral for $n = 0, 1$ (and all odd n), and its degree of precision is 1. (One could also observe that this is trapezoid quadrature on $[-1, 1]$, which is known to be exact for polynomials up to degree 1.)

(b) 2/3[f(-1) + f(0) + f(1)]. We check the quadrature on monomials of increasing degree:

$$2/3[(-1)^n + 0^n + (-1)^n] = \begin{cases} 2, & \text{for } n = 0, \\ 0, & \text{for odd } n, \\ 4/3, & \text{for even } n > 0, \end{cases}$$

which agrees with the integral for $n = 0, 1$ (and odd n). Thus degree of precision is 1.

(c) We check the quadrature on monomials of increasing degree:

$$[(-1/\sqrt{3})^n + (1/\sqrt{3})^n] = \begin{cases} 0, & \text{for odd } n, \\ 2/3^{n/2}, & \text{for even } n, \end{cases}$$

which agrees with the integral for $n = 0, 1, 2, 3$ (and odd n). Thus degree of precision is 3. (This is an example of Gaussian quadrature, which are exact for polynomials of degree up to $2n - 1$ when n points are used.)

3 Exercise 5.2.16

Use the fact that the error term of Boole's Rule (see Exercise 5.2.15) is proportional to $f^{(6)}(c)$ to find the exact error term.

Solution

We apply the quadrature to the monomial x^6, for which $f^{(6)}(c) \equiv 6!$, for all c. Thus

$$\int_0^{4h} x^6 \,\mathrm{d}x = \left[\frac{x^7}{7}\right]_{x=0}^{x=4h} = \frac{4^7 h^7}{7} = \frac{2h}{45}(7 \cdot 0^6 + 32 \cdot h^6 + 12 \cdot (2h)^6 + 32 \cdot (3h)^6 + 7 \cdot (4h)^6) + C6!$$

and we are interested in finding C.

In [1]:
```
from sympy import *
init_printing()
h,C=symbols('h C')
integral = (4*h)**7/7
quadrature = 2*h/45*(32*h**6 + 12*(2*h)**6 + 32*(3*h)**6 + 7*(4*h)**6)
simplify(integral-quadrature)
```

Out[1]:

$$-\frac{128h^7}{21}$$

Thus $C = -128h^7/(21 \cdot 6!) = -8h^7/945$

4 Exercise 5.4.1

Apply Adaptive Quadrature by hand, using the Trapezoid Rule with tolerance $TOL = 0.05$ to approximate the integrals. Find the approximation error.

Solution

(a) In this case, $f(x) = x^2$, $a_0 = 0$, $b_0 = 1$.

We begin with $n = 1$ interval, $a = a_0$, $b = b_0$. We will use $S[a, b]$ to denote the Trapezoid quadrature applied on the interval $[a, b]$.

$c = (a+b)/2$, $S[a, b] = (0^2+1^2)/2 = 1/2$, $S[a, c] = (0^2+(1/2)^2)/4 = 1/16$, $S[c, b] = ((1/2)^2+1^2)/4 = 5/16$. $|S[a, b] - S[a, c] - S[c, b]| = 1/8 = 0.125 < 3 \cdot 0.05 \cdot ((b - a)/(b_0 - a_0)) = 0.15$. Thus we stop with the approximation $S[a, c] + S[c, b] = 3/8 = 0.375$ and an error estimate $-(S[a, b] - S[a, c] - S[c, b])/3 = -1/24 \approx -0.0417$.

The actual integration error is $\int_0^1 x^2 - 3/8 = 1/3 - 3/8 = -1/24$. Our estimate is exact because f'' is constant in this case, thus the approximate equation (5.38) in the book is actually exact.

(b) $f(x) = \cos(x)$, $a_0 = 0$, $b_0 = \pi/2$.

$S[0, \pi/2] = (1 + 0)/2 = 1/2$, $S[0, \pi/4] = (1 + 1/\sqrt{2})/4 \approx 0.42678$, $S[\pi/4, \pi, 2] = (1/\sqrt{2} + 0)/4 \approx 0.17678$. $|S[a, b]-S[a, c]-S[c, b]| = |1/4-1/(2\sqrt{2})| \approx 0.104 < 3\cdot 0.05\cdot 1 = 0.15$. Thus we stop with the approximation $S[a, c] + S[c, b] = 1/4 + 1/(2\sqrt{2}) \approx 0.60355$ and an error estimate $-(S[a, b] - S[a, c] - S[c, b])/3 = 1/(6\sqrt{2}) - 1/12 \approx 0.0345$, whereas the actual integration error is $\int_0^{\pi/2} \cos(x)-1/4-1/(2\sqrt{2}) = 1-1/4-1/(2\sqrt{2}) \approx 0.396$.

Thus the adaptive quadrature underestimates the error in this case. The problem is the assumption made in equation (5.38) is not realistic in this case.

(c) In this case, $f(x) = \exp(x)$, $a_0 = 0$, $b_0 = 1$.

$S[0, 1] = (1 + e)/2 \approx 1.85914$, $S[0, 0.5] = (1 + \exp(0.5))/4 \approx 0.66218$, $S[0.5, 1] = (\exp(0.5) + \exp(1))/4 \approx 1.09175$. $|S[a, b] - S[a, c] - S[c, b]| \approx 0.1052 < 3 \cdot 0.05 \cdot ((b - a)/(b_0 - a_0)) = 0.15$. Thus we stop with the approximation $S[a, c] + S[c, b] \approx 1.75393$ and an error estimate $-(S[a, b] - S[a, c] - S[c, b])/3 =\approx -0.035$.

The actual integration error is $\int_0^1 \exp(x) - (S[a, c] + S[c, b]) \approx -0.036$. Our estimate is quite good in this case.

5 Exercise 5.4.4

Develop an Adaptive Quadrature method for rule (5.28).

Solution

Let $c = (a + b)/2$, $h = b - a$ and apply the quadrature on $[a, b]$, $[a, c]$, and $[c, b]$ (we use the same notation as in the book):

$$\int_a^b f(x)\,\mathrm{d}x = S[a,b] + \frac{14h^5}{45} f^{(4)}(c_1), \qquad c_1 \in [a,b],$$

$$\int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x = S[a,c] + S[c,b] + \frac{14(h/2)^5}{45} f^{(4)}(c_2) + \frac{14(h/2)^5}{45} f^{(4)}(c_3), \qquad c_2 \in [a,c], c3 \in [c,b].$$

Further assuming that $f^{(4)}(c_1) \approx f^{(4)}(c_2) \approx f^{(4)}(c_3)$ we obtain:

$$\int_a^b f(x)\,\mathrm{d}x = S[a,b] + \frac{14h^5}{45} f^{(4)}(c_1),$$

$$\int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x \approx S[a,c] + S[c,b] + \frac{1}{16}\frac{14h^5}{45} f^{(4)}(c_1).$$

We can now subtract the second equation from the first to obtain:

$$S[a,b] - S[a,c] - S[c,b] \approx 15\frac{1}{16}\frac{14h^5}{45} f^{(4)}(c_1).$$

Thus the number $S[a, b] - S[a, c] - S[c, b]$ gives an approximation to 15 times the error of the quadrature $S[a, c] + S[c, b]$.

The rest is exactly the same as for other quadratures; one stops subdividing the interval when $|S[a,b] - S[a,c] - S[c,b]| < 15 \cdot TOL \cdot (b-a)/(b_{\text{orig}} - a_{\text{orig}})$ and returns $S[a,c] + S[c,b]$ as the approximation of the integral over the interval [a,b].

6 Exercise 5.5.1

Approximate the integrals using $n = 2$ Gaussian Quadrature. Compare with the correct value and give the approximation error.

Solution

Gaussian Quadrature with $n = 2$ is defined by the points $x_{1,2} = \pm 1/\sqrt{3}$, $w_{1,2} = 1$.

In [2]:
```
x1 = -1/sqrt(3)
x2 =  1/sqrt(3)
w1 =  1
w2 =  1
```

(a)

In [3]:
```
from sympy import *
init_printing()
x=symbols('x')
f=Lambda(x,x**3+2*x)

# Exact integral
integrate(f(x),(x,-1,1))
```

Out[3]:

$$0$$

In [4]:
```
# Numerical quadrature
f(x1)*w1 + f(x2)*w2
```

Out[4]:

$$0$$

Thus the error is 0, which agrees with the theory for Gaussian Quadratures being exact for polynomials up to degree $2n - 1$.

(b)

In [5]:
```
f=Lambda(x,x**4)

# Exact integral
integrate(f(x),(x,-1,1))
```

Out[5]:

$$\frac{2}{5}$$

In [6]:
```
# Numerical quadrature
f(x1)*w1 + f(x2)*w2
```

Out[6]:

$$\frac{2}{9}$$

In [7]:
```
# Approximation error
2/5-2/9
```

Out[7]:

$$0.17777777777777778$$

(c)

In [8]:
```
f=Lambda(x,exp(x))

# Exact integral
integrate(f(x),(x,-1,1))
```

Out[8]:

$$-\frac{1}{e} + e$$

In [9]:
```
# Numerical quadrature
f(x1)*w1 + f(x2)*w2
```

Out[9]:

$$e^{-\frac{\sqrt{3}}{3}} + e^{\frac{\sqrt{3}}{3}}$$

In [10]:
```
# Approximation error
N(integrate(f(x),(x,-1,1))-(f(x1)*w1 + f(x2)*w2))
```

Out[10]:

$$0.00770629937787234$$

(d)

In [11]:
```
f=Lambda(x,cos(pi*x))

# Exact integral
integrate(f(x),(x,-1,1))
```

Out[11]:

$$0$$

In [12]:
```
# Numerical quadrature
f(x1)*w1 + f(x2)*w2
```

Out[12]:

$$2\cos\left(\frac{\sqrt{3}\pi}{3}\right)$$

In [13]:
```
# Approximation error
N(0-2*cos(sqrt(3)*pi/3))
```

Out[13]:

$$0.481237029038818$$

7 Exercise 5.5.7

Show that the Legendre polynomials $p_1(x) = x$ and $p_2(x) = x^2 - 1/3$ are orthogonal on $[-1, 1]$.

Solution $\int_{-1}^{1} p_1(x)p_2(x)\,\mathrm{d}x = \int_{-1}^{1}(x^3 - x/3)\,\mathrm{d}x = 0$ because $x^3 - x/3$ is an odd function.

8 Exercise 5.5.8

Compute Legendre polynomials up to degree 3.

Solution

Legendre polynomials are given by the expression

$$p_i(x) = \frac{1}{2^i i!}\frac{\mathrm{d}}{\mathrm{d}x^i}[(x^2-1)^i].$$

Thus

$$\begin{aligned}
&p_0(x) = 1,\\
&p_1(x) = 1/2[(x^2-1)]' = x,\\
&p_2(x) = 1/(4\cdot 2)[(x^2-1)^2]'' = 1/8[x^4 - 2x^2 + 1]'' = 1/8[12x^2 - 4] = 3/2(x^2 - 3),\\
&p_3(x) = 1/(8\cdot 6)[(x^2-1)^3]''' = 1/48[x^6 - 3x^4 + 3x^2 - 1]''' = 1/48[120x^3 - 72x] = 5/2[x^3 - 3/5x].
\end{aligned}$$

9 Exercise 5.5.9

Verify the coefficients c_i and x_i in Table 5.1 for degree $n = 3$.

Solution

We have computed $p_3(x) = 5/2x(x^2 - 3/5)$ in Exercise 5.5.8. This immediatly implies that $p_3(0) = p_3(\pm\sqrt{3/5}) = 0$ confirming the roots listed in Table 5.1.

The coefficients c_i are obtained by integrating the Lagrange polynomials. For example,

$$L_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} = \frac{(x+\sqrt{3/5})(x-\sqrt{3/5})}{(0+\sqrt{3/5})(0-\sqrt{3/5})} = 1 - \frac{5}{3}x^2,$$

and therefore

$$c_2 = \int_{-1}^{1}[1 - 5/3x^2]\,\mathrm{d}x = [x - 5/9x^3]_{x=-1}^{x=1} = 2 - 10/9 = 8/9.$$

Similarly

$$L_{1,3}(x) = \frac{(x-0)(x\mp\sqrt{3/5})}{(\mp\sqrt{3/5}-0)(\mp\sqrt{3/5}\mp\sqrt{3/5})} = \frac{x(x\mp\sqrt{3/5})}{2\cdot 3/5},$$

and therefore

$$c_{1,3} = \frac{5}{6}\int_{-1}^{1}[x^2 \mp x\sqrt{3/5}]\,\mathrm{d}x = \frac{5}{6}[x^3/3 \mp \sqrt{3/5}x^2/2]_{x=-1}^{x=1} = \frac{5}{6}\frac{2}{3} = \frac{5}{9}.$$

10 Computer exercise 5.2.8

Answers:

(a) 1.547866

(b) 1.277978

(c) 1.277978

11 Computer exercise 5.2.10

See the file uniform_refinement.m available on the wiki. Adapt the code to use Simpson's rule instead of Trapezoid.