Solutions_2

January 23, 2016

1 Exercise 3.1.1 (a), p. 149

Let us compute Lagrange polynomials first for our case $x_1 = 0$, $x_2 = 2$, $x_3 = 3$:

```
In [14]: from sympy import *
         init_printing(use_latex=True)
         x=symbols('x')

         L1 = (x-2)*(x-3)/((0-2)*(0-3))
         L2 = (x-0)*(x-3)/((2-0)*(2-3))
         L3 = (x-0)*(x-2)/((3-0)*(3-2))
```

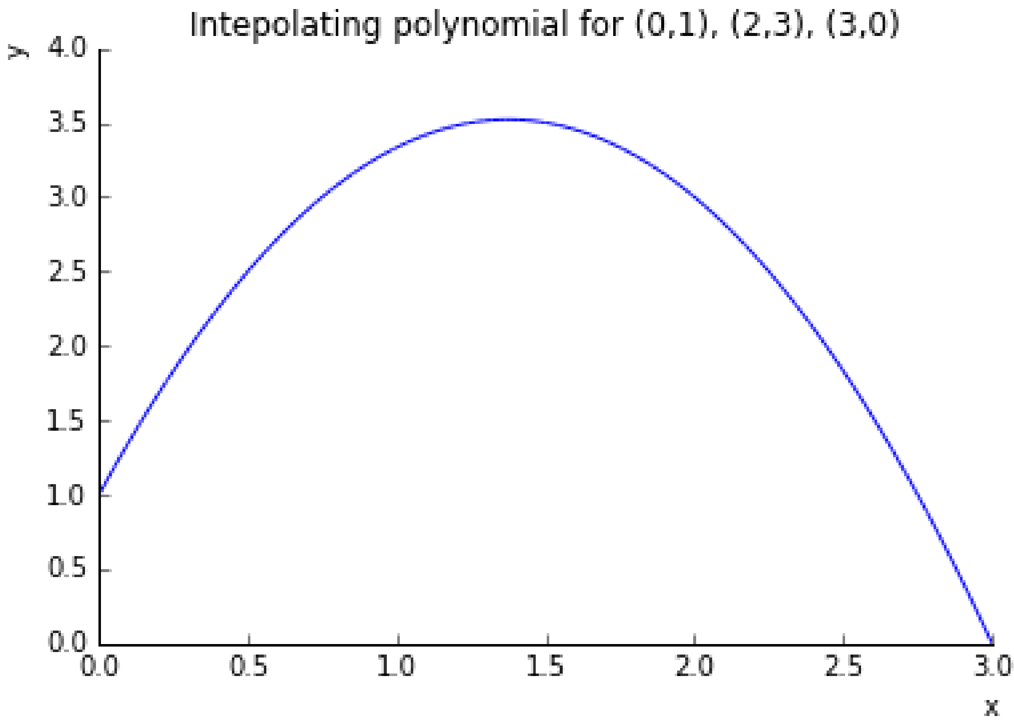
Now the interpolating polynomial for $y_1 = 1$, $y_2 = 3$, $y_3 = 0$:

```
In [15]: p = 1*L1 + 3*L2 + 0*L3
         expand(p)
```

Out[15]:

$$-\frac{4x^2}{3} + \frac{11x}{3} + 1$$

```
In [16]: from sympy.plotting import plot
         %matplotlib inline
         plot(p,(x,0,3),xlabel='x',ylabel='y',title='Intepolating polynomial for (0,1), (2,3), (3,0)')
```



Out[16]: <sympy.plotting.plot.Plot at 0x106013ef0>

2 Exercise 3.1.1 (b), p. 149

$x_1 = -1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 5$:

In [9]:
```
L1 = (x-2)*(x-3)*(x-5)/((-1-2)*(-1-3)*(-1-5))
L2 = (x+1)*(x-3)*(x-5)/((2+1)*(2-3)*(2-5))
L3 = (x+1)*(x-2)*(x-5)/((3+1)*(3-2)*(3-5))
L4 = (x+1)*(x-2)*(x-3)/((5+1)*(5-2)*(5-3))
```

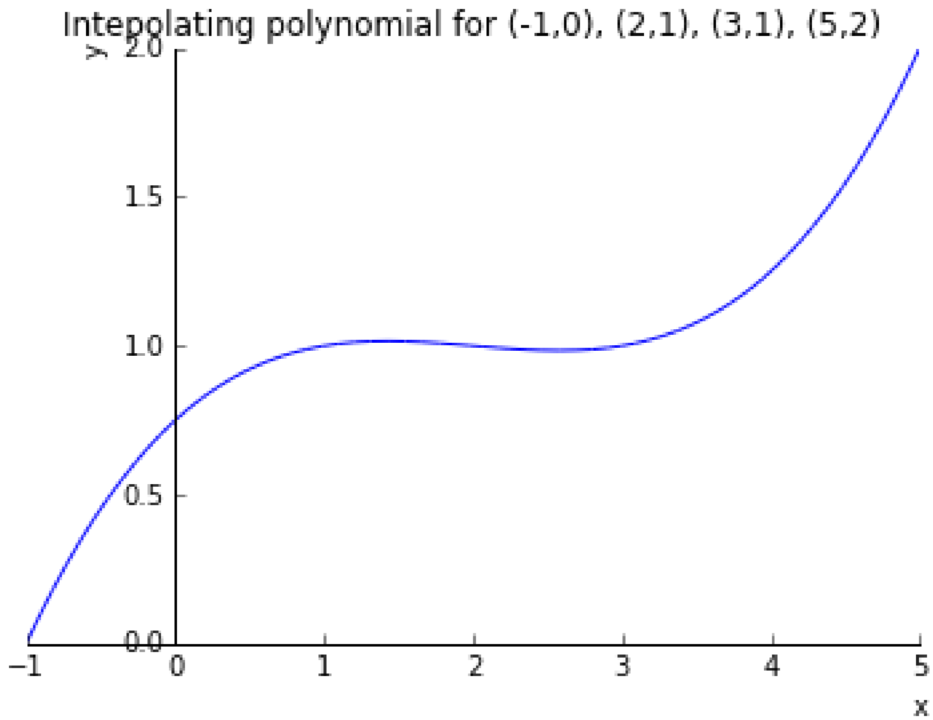
Now the interpolating polynomial for $y_1 = 0$, $y_2 = 1$, $y_3 = 1$, $y_4 = 2$:

In [10]:
```
p=0*L1 + 1*L2 + 1*L3 + 2*L4
expand(p)
```

Out[10]:

$$\frac{x^3}{24} - \frac{x^2}{4} + \frac{11x}{24} + \frac{3}{4}$$

In [12]:
```
plot(p,(x,-1,5),xlabel='x',ylabel='y',title='Intepolating polynomial for (-1,0), (2,1), (3,1),
```



Out[12]: <sympy.plotting.plot.Plot at 0x10719c5f8>

3 Exercise 3.1.1 (c), p. 149

$x_1 = 0$, $x_2 = 2$, $x_3 = 4$:

In [17]:
```
L1 = (x-2)*(x-4)/((0-2)*(0-4))
L2 = (x-0)*(x-4)/((2-0)*(2-4))
L3 = (x-0)*(x-2)/((4-0)*(4-2))
```

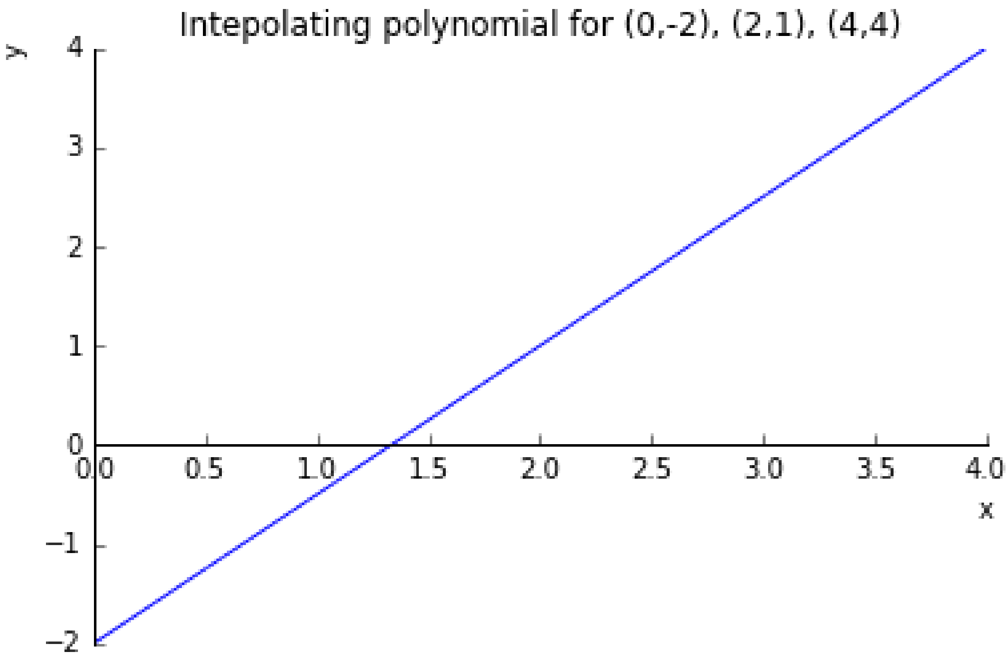
Now the interpolating polynomial for $y_1 = -2$, $y_2 = 1$, $y_3 = 4$:

In [18]:
```
p=-2*L1 + 1*L2 + 4*L3
expand(p)
```

Out[18]:

$$\frac{3x}{2} - 2$$

In [19]:
```
plot(p,(x,0,4),xlabel='x',ylabel='y',title='Intepolating polynomial for (0,-2), (2,1), (4,4)')
```




```
Out[19]: <sympy.plotting.plot.Plot at 0x1074cccc0>
```

4 Exercise 3.1.2 (a), p. 149

Let us compute Newton's divided differences in this case:

$$f[x_1] = y_1 = 1 \qquad f[x_2] = y_2 = 3 \qquad f[x_3] = y_3 = 0$$
$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{3-1}{2-0} = 1 \qquad f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2} = \frac{0-3}{3-2} = -3$$
$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} = \frac{-3-1}{3-0} = -\frac{4}{3}$$

Therefore, the final polynomial is:

$$p(x) = f[x_1] + f[x_1, x_2](x-x_1) + f[x_1, x_2, x_3](x-x_1)(x-x_2) = 1 + 1\cdot(x-0) - \frac{4}{3}\cdot(x-0)(x-2) = 1 + x - \frac{4}{3}x^2 + \frac{8}{3}x = -\frac{4}{3}x^2 + \frac{11}{3}x + 1$$

5 Exercise 3.1.2 (b), p. 149

$$f[x_1] = y_1 = 0 \qquad f[x_2] = y_2 = 1 \qquad f[x_3] = y_3 = 1 \qquad f[x_4] = y_4 = 2$$
$$f[x_1, x_2] = \frac{1-0}{2+1} = \frac{1}{3} \qquad f[x_2, x_3] = \frac{1-1}{3-2} = 0 \qquad f[x_3, x_4] = \frac{2-1}{5-3} = \frac{1}{2}$$
$$f[x_1, x_2, x_3] = \frac{0 - 1/3}{3+1} = -\frac{1}{12} \qquad f[x_2, x_3, x_4] = \frac{1/2 - 0}{5-2} = \frac{1}{6}$$
$$f[x_1, x_2, x_3, x_4] = \frac{1/6 + 1/12}{5+1} = \frac{1}{24}$$

Therefore, the interpolating polynomial is

```
In [8]: expand(0+Rational(1,3)*(x+1)-Rational(1,12)*(x+1)*(x-2)+Rational(1,24)*(x+1)*(x-2)*(x-3))
```

Out[8]:

$$\frac{x^3}{24} - \frac{x^2}{4} + \frac{11x}{24} + \frac{3}{4}$$

6 Exercise 3.1.2 (c), p.149

$f[x_1] = -2 \qquad f[x_2] = 1 \qquad f[x_3] = 4 f[x_1, x_2] = \frac{1+2}{2-0} = \frac{3}{2} \qquad f[x_2, x_3] = \frac{4-1}{4-2} = \frac{3}{2} f[x_1, x_2, x_3] = \frac{3/2 - 3/2}{4-0} = 0$

Threefore, the interpolating polynomial is

$$p(x) = f[x_1] + f[x_1, x_2](x - x_1) + f[x_1, x_2, x_3](x - x_1)(x - x_2) = -2 + \frac{3}{2}(x - 0) = \frac{3}{2}x - 2.$$

7 Exercise 3.1.11, p. 150

Let $P_1, \dots, P_4$ be four different points lying on a parabola $y = ax^2 + bx + c$. How many degree 3 polynomials pass through these four points?

Solution There is a unique polynomial of degree ≤ 3 passing through four different points. Therefore this polynomial must be the given parabola. In other words, there are no cubic polynomials passing through these points, only a quadratic one.

8 Exercise 3.1.12, p. 150

Can a degree 3 polynomial intersect a degree 4 polynomial in exactly five points?

Solution Again, owing to the uniqueness of the interpolation polynomial there is only one polynomial of degree ≤ 4 (that is, constant, linear, quadratic, cubic, or quartic) passing through five different points. Therefore it is impossible to both cubic and quartic polynomials to pass through some five points. (intersect at exactly five points).

9 Exercise 3.1.15, p. 150

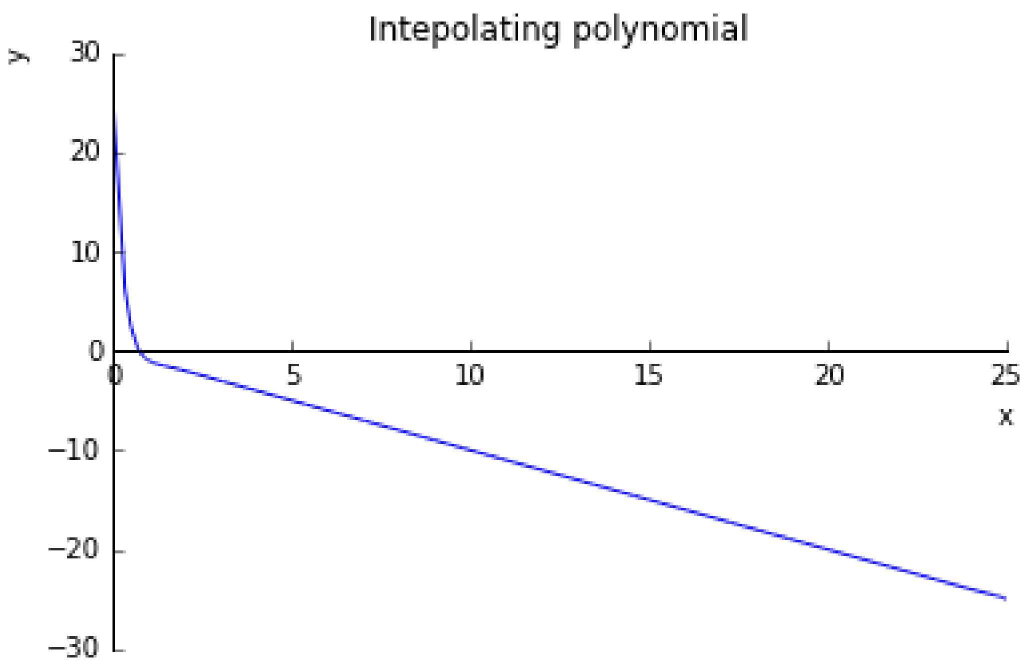
Write a degree 25 polynomial that passes through the points $(1, -1), (2, -2), \dots, (25, -25)$ and have a constant term equal to 25.

Solution It is not difficult to see that a linear polynomial $-x$ passes through the given points $(1, -1), (2, -2), \dots, (25, -25)$. We can then use Newton's form of the interpolation polynomial to add another point, namely $x = 0$, where the function should attain the value $y = 25$. Thus

$$p(x) = -x + f[1, \dots, 25, 0](x - 1)(x - 2) \dots (x - 25)$$

We can now use the condition $p(0) = 25$ to determine the constant $f[1, \dots, 25, 0]$. Substituting $x = 0$ into the formula for our polynomial gives $25 = p(0) = f[1, \dots, 25, 0](-1)^{25} 25! = -f[1, \dots, 25, 0]25!$, and therefore the final formula is $p(x) = -x - (x - 1)(x - 2) \dots (x - 25)/24!$.

```
In [32]: from sympy.mpmath import *
         from sympy.plotting import plot
         %matplotlib inline
         p=-x-nprod(lambda k: (x-k),[1,25])/fac(24)
         plot(p,(x,0,25),xlabel='x',ylabel='y',title='Intepolating polynomial')
```



Out[32]: <sympy.plotting.plot.Plot at 0x107443160>

10 Exercise 3.2.2, p.156

(a) Given the data points $(1, 0), (2, \ln 2), (4, \ln 4)$ find the degree 2 interpolating polynomial.
Solution Newton's divided differences are:

$$f[1] = 0 \qquad f[2] = \ln 2 \qquad f[4] = \ln 4$$
$$f[1,2] = \ln 2 \qquad f[2,3] = \frac{\ln 4 - \ln 2}{4-2} = \frac{\ln 2}{2}$$
$$f[1,2,3] = \frac{\ln 2/2 - \ln 2}{4-1} = -\frac{\ln 2}{6}$$

and therefore the interpolating polynomial is

$$p(x) = 0 + \ln 2(x-1) - \frac{\ln 2}{6}(x-1)(x-2) = \left[-\frac{1}{6}x^2 + \frac{3}{2}x - \frac{4}{3}\right]\ln 2.$$

(b) Use the result of (a) to approximate $\ln 3$:
Solution

$$p(3) = \frac{5}{3}\ln 2 \approx 1.1552$$
$$\ln(3) \approx 1.0986$$

(c) Use Theorem 3.3 to give an error bound for the approximation in part (b)
Solution The error bound is

$$|\ln 3 - p(3)| = \left|\frac{\ln'''(z)}{3!}(3-1)(3-2)(3-4)\right| = \frac{2}{6z^3}2\cdot 1\cdot 1 = \frac{2}{3z^3},$$

for some $z \in [1, 4]$. The worst (largest) estimate is when $z = 1$ leading to

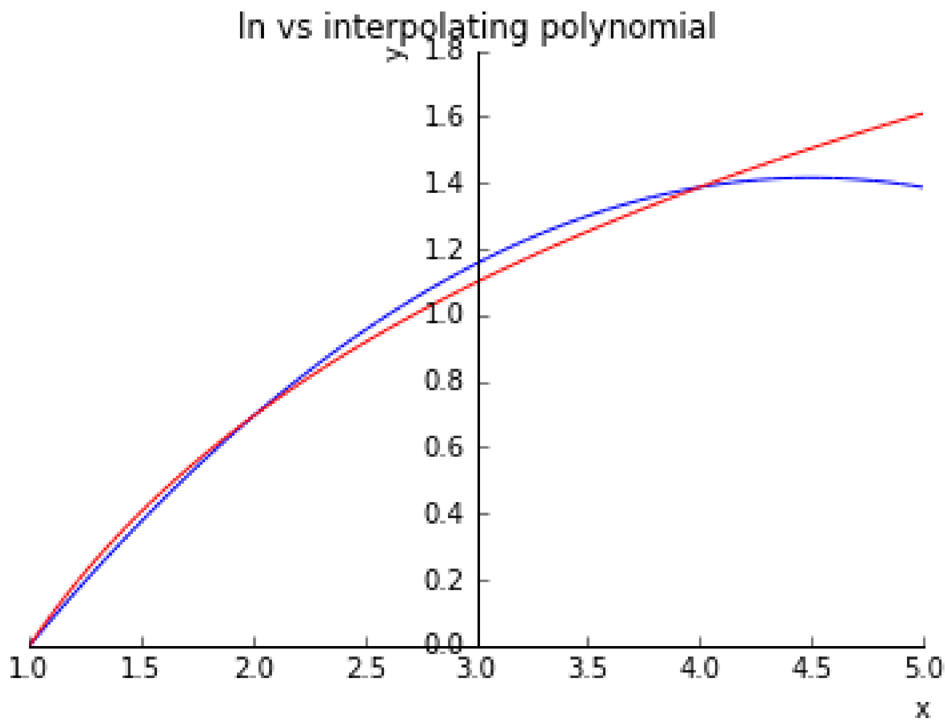
$$|\ln 3 - p(3)| \leq \frac{2}{3}.$$

(d) Compare with the actual error.
Solution The actual error is

$$|\ln(3) - p(3)| \approx 0.0566 < \frac{2}{3} \approx 0.667.$$

Thus we overestimate the error by a factor of almost 12. This is because we do not know the actual point on our interval, at which $\ln'''(z)$ should be evaluated.

```
In [27]: f = log(x)
         p = (-Rational(1,6)*x**2+Rational(3,2)*x-Rational(4,3))*log(2)
         plot1 = plot(f,(x,1,5),line_color='r',show=False)
         plot2 = plot(p,(x,1,5),xlabel='x',ylabel='y',title='ln vs interpolating polynomial',line_color=
         plot2.extend(plot1)
         plot2.show()
```



11 Exercise 3.2.4, p.156

Consider the interpolating polynomial for $f(x) = 1/(x+5)$ with interpolation nodes $x = 0, 2, 4, 6, 8, 10$. Find an upper bound for the interpolation error at (a) $x = 1$ and (b) $x = 5$.

Solution The upper bound on the interpolation error is given by Theorem 3.3:

$$|f(x) - p(x)| = \left| \frac{f^{(6)}(z)}{6!}(x - x_1) \ldots (x - x_6) \right|,$$

for some $z \in [0, 10]$. $|f^{(6)}(z)| = 6!/(z+5)^7$ with the largest (pessimistic) value occurring at $z = 0$, when $|f^{(6)}(0)| = 6!/5^7$.

(a) Here we get

$$|f(1) - p(1)| \leq \frac{1}{5^7}(1-0)(2-1)(4-1)(6-1)(8-1)(10-1) \approx 0.012$$

(b) Here we get

$$|f(5) - p(5)| \leq \frac{1}{5^7}(5-0)(5-2)(5-4)(6-5)(8-5)(10-5) \approx 0.0029$$

Thus the estimate for the error in the middle is significantly (by a factor of 4.2) smaller.

12 Exercise 3.3.3, p. 165

Assume that Chebyshev interpolation is used to find a fifth degree interpolating polynomial $Q_5(x)$ on the interval $[-1, 1]$ for the function $f(x) = e^x$. Use the interpolation error formula to find a worst case estimate for the error $|e^x - Q_5(x)|$ that is valid for x throughout the interval $[-1, 1]$. How many digits after the decimal point will be correct when $Q_5(x)$ is used to approximate e^x?

Solution Fifth degree interpolation polynomial implies that at least $n = 6$ interpolation nodes are used. Using the interpolation error formula for Chebyshev polynomial on $[-1, 1]$ we get

$$\max_{x\in[-1,1]} |f(x) - Q_5(x)| \leq \max_{z\in[-1,1]} \frac{|f^{(6)}(z)|}{2^{6-1}6!} = \max_{z\in[-1,1]} \frac{e^z}{2^5 6!} = \frac{e}{2^5 6!} \approx 1.2 \cdot 10^{-4}.$$

Thus at least three digits after the decimal point will be correct.

13 Exercise 3.3.7, p.165

Suppose that we are designing the ln key for a calculator whose display shows six digits to the right of the decimal point. Find the least degree d for which Chebyshev interpolation on the interval $[1, e]$ will approximate within this accuracy.

Solution We use formulae (3.6) and (3.14) in the book. Thus for a Chebyshev interpolation polynomial of degree d ($d+1$ interpolation nodes) we obtain:

$$\max_{x\in[1,e]} |f(x) - Q_d(x)| \leq \max_{z\in[1,e]} \frac{|f^{d+1}(z)|}{(d+1)!2^d} \frac{(e-1)^{d+1}}{2^{d+1}} = \max_{z\in[1,e]} \frac{d!}{z^{d+1}(d+1)!2^d} \frac{(e-1)^{d+1}}{2^{d+1}} = \frac{(e-1)^{d+1}}{(d+1)2^{2d+1}},$$

because the maximum of $1/z$ on $[1, e]$ occurs at $z = 1$. We now look for an integer d such that the error is strictly smaller than 10^{-6}:

$$\begin{aligned} d &= 1 && \text{error} \leq 0.18 \\ d &= 2 && \text{error} \leq 0.053 \\ d &= 6 && \text{error} \leq 7.7 \cdot 10^{-4} \\ d &= 10 && \text{error} \leq 1.7 \cdot 10^{-5} \\ d &= 13 && \text{error} \leq 1.04 \cdot 10^{-6} \\ d &= 14 && \text{error} \leq 4.17 \cdot 10^{-7} \end{aligned}$$

Thus $d = 14$ is required.