NTNU

Kunnskap for en bedre verden

Institutt for matematiske fag

Eksamensoppgave i
TMA4320 Introduksjon til vitenskapelige beregninger

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Eksamenstid (fra–til): 09:00–13:00

Hjelpemiddelkode/Tillatte hjelpemidler: B: Spesifiserte trykte og håndskrevne hjelpemidler tillatt:

- K. Rottmann: Matematisk formelsamling

Bestemt, enkel kalkulator tillatt.

Målform/språk: bokmål

Antall sider: 7

Antall sider vedlegg: 0

Kontrollert av:

Dato Sign

Oppgave 1 Vi ser på likningen

$$e^x - y = 0,$$

hvor $y > 0$ er gitt, og $x \in \mathbb{R}$ er ukjent.

a) Formuler Newtons metode for å løse denne likningen. Gjør to iterasjoner for hånd for $y = e$. Start med $x_0 = 0$.

Solution: direct computation. The Newton's iteration for the equation $f(x) = 0$ is $x_{k+1} = x_k - f(x_k)/f'(x_k)$, which in our case simplifies to

$$x_{k+1} = x_k - (e^{x_k} - y)/e^{x_k} = x_k - 1 + ye^{-x_k}.$$

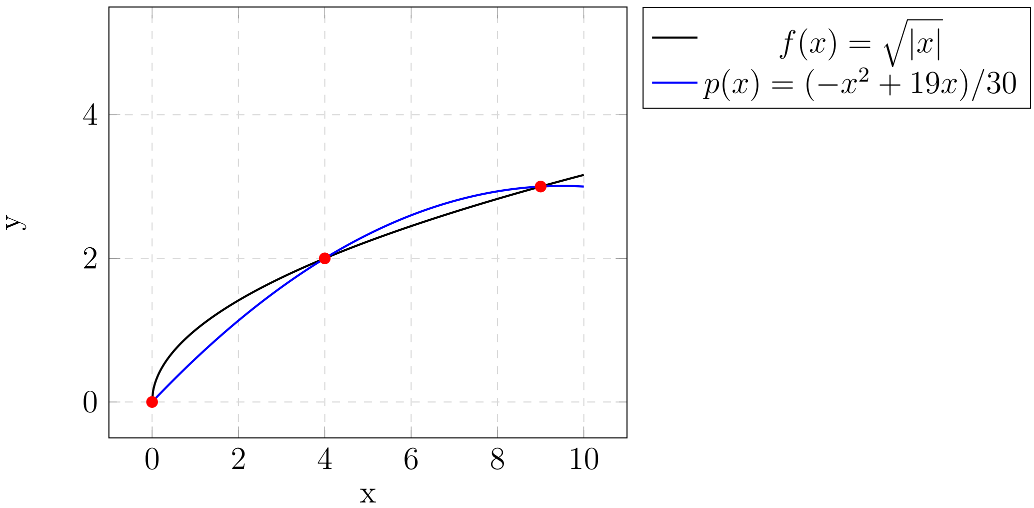
Thus $x_0 = 0$, $x_1 \approx 1.718281828459045$, $x_2 \approx 1.205871127178306$

Oppgave 2

a) Finn det polynomet $p(x)$ av *lavest mulig grad* som interpolerer funksjonen $f(x) = \sqrt{|x|}$ i punktene $x_1 = 0$, $x_2 = 4$, $x_3 = 9$.

Solution: direct computation. For example using Lagrange's form of the interpolation polynomial we get

$$\begin{aligned}
L_1(x) &= \frac{(x-4)(x-9)}{(0-4)(0-9)} = \frac{x^2 - 13x + 36}{36} \\
L_2(x) &= \frac{(x-0)(x-9)}{(4-0)(4-9)} = -\frac{x^2 - 9x}{20} \\
L_3(x) &= \frac{(x-0)(x-4)}{(9-0)(9-4)} = \frac{x^2 - 4x}{45} \\
P(x) &= \sqrt{0}L_1(x) + \sqrt{4}L_2(x) + \sqrt{9}L_3(x) = -\frac{x^2 - 9x}{10} + \frac{x^2 - 4x}{15} \\
&= \frac{-3x^2 + 27x + 2x^2 - 8x}{30} = \frac{-x^2 + 19x}{30}
\end{aligned}$$



Vi bruker samme notasjon for $f(x)$ og $p(x)$ som i **a)** i resten av oppgaven.

b) Funksjonen $F(x) = x$ er lik med $(f(x))^2$ for $x \geq 0$. Derfor interpolerer $P(x) = (p(x))^2$ funksjonen $F(x)$ i punktene $x_1 = 0$, $x_2 = 4$ og $x_3 = 9$.

Forklar hvorfor formelen for estimatet av interpolasjonsfeilen, gitt av

$$F(x) - P(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{3!} F'''(c),$$

med $c \in [\min\{x, x_1\}, \max\{x, x_3\}]$, *ikke* holder i denne situasjonen.

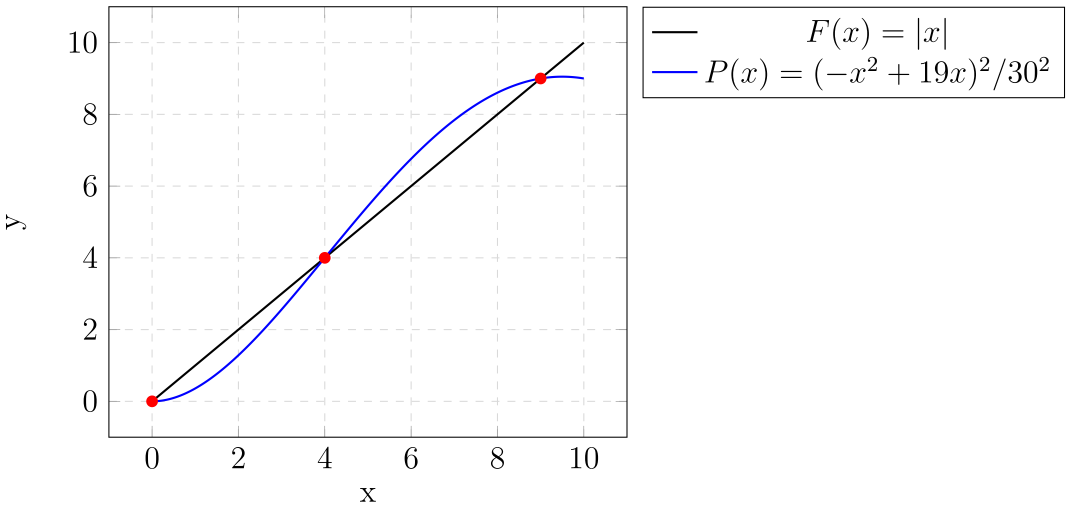
Solution: The short answer here is that the error estimate is valid only for the interpolation polynomials of minimal degree, whereas $P(x) = (p(x))^2$ is not an interpolation polynomial of minimal degree. (Indeed, it has degree 4, whereas for 3 interpolation nodes we should have degree 2 or less. In fact in the present situation the interpolation polynomial is $F(x) = x$, and thus has degree 1.)

One could see that the error estimate predicts that the interpolation error should be zero (as $F'''(x) = (x)''' = 0$) and it is for the interpolation polynomial of minimal degree, that is x, but clearly not for our interpolation polynomial $P(x)$.

Why is the error estimate

$$F(x) - P(x) = \frac{(x - x_1)(x - x_2) \dots (x - x_n)}{n!} F^{(n)}(c),$$

only valid for polynomials of minimal degree? Suppose we remove the minimal degree requirement. Then we have n interpolation points and we look at interpolating polynomials P of degree at least n. With polynomials of this degree we can interpolate through $n+1$ points. Thus we can chose $n+1$-st interpolation node x_{n+1} arbitrarily and place the corresponding $y = P(x_{n+1})$-value as far from $F(x_{n+1})$ as we like. Then the left hand side of the estimate, $F(x_{n+1}) - P(x_{n+1})$ can be made arbitrarily large, whereas the right hand side of the estimate is determined by the position of the nodes $x_1, \ldots, x_n, x_{n+1}$ and the n-th derivative of F.



Oppgave 3

a) Approksimer integralet $\int_0^1 \ln(x)\,\mathrm{d}x$ ved å bruke midtpunktskvadraturer med $n = 1$ og $n = 2$ delintervaller.[1]

Solution: Direct computation. With 1 panel we have:

$$\int_0^1 \ln(x)\,\mathrm{d}x \approx 1\ln(0.5) \approx -0.6931471805599453$$

With 2 panels we have:

$$\int_0^1 \ln(x)\,\mathrm{d}x \approx 0.5[\ln(0.25) + \ln(0.75)] \approx -0.8369882167858358.$$

b) Hvis vi antar at f'' er kontinuerlig på intervallet $[a, b]$, er feilestimatet for midtpunktskvadraturet $Q_{[a,b]}f$ gitt av

$$\int_a^b f(x)\,\mathrm{d}x = Q_{[a,b]}f + \frac{h^3}{24} f''(c),$$

[1] delintervaller = “panels” i boken

der c er et punkt mellom a og b, og $h = b - a$.

Bruk nå adaptive kvadraturer til å estimere forskjellen $\int_0^1 \ln(x)\,\mathrm{d}x - Q_{[0,1]} \ln$. (Ignorer det at ln på intervallet $[0,1]$ ikke oppfyller deriverbarhetskravet i feilestimatet.) Du kan gjenbruke de numeriske beregningene fra **a)**.

Solution: We have:

$$0 = Q_{[0,1]} \ln - (Q_{[0,0.5]} \ln + Q_{[0.5,1]} \ln) + \frac{1^3}{24} \ln''(c_1) - \frac{0.5^3}{24} \ln''(c_2) - \frac{0.5^3}{24} \ln''(c_3).$$

We assume that $\ln''(c_1) \approx \ln''(c_2) \approx \ln''(c_3)$ (this is the standard assumption in adaptive quadratures) and thus get

$$\begin{aligned} \frac{0.75}{24} \ln''(c_1) &= (Q_{[0,0.5]} \ln + Q_{[0.5,1]} \ln) - Q_{[0,1]} \ln \\ &\approx -0.8369882167858358 - (-0.6931471805599453) \\ &\approx -0.1438410362258905. \end{aligned}$$

Thus the error estimate is

$$\begin{aligned} \int_0^1 \ln(x)\,\mathrm{d}x - Q_{[0,1]} \ln = \frac{1}{24} \ln''(c_1) &\approx 4/3 * (-0.1438410362258905) \\ &\approx -0.19178804830118734.^{2} \end{aligned}$$

Oppgave 4 Vi skal nå se på et initialverdiproblem:

$$y'(t) = -(y(t))^2, \qquad y(0) = 1.$$

Løsningen av differensiallikningen er $y(t) = (t+1)^{-1}$.

a) Regn ut to steg av y numerisk ved hjelp av den eksplisitte Eulermetoden. Bruk steglengde $h = 1$.

Solution: Direct computation. Explicit Eulers method for this IVP is

$$w_{i+1} = w_i - h w_i^2 = w_i(1 - h w_i),$$

$w_0 = y(0) = 1$.

Thus we get:

$$\begin{aligned} w_1 &= w_0(1 - h w_0) = 1(1 - 1 \cdot 1) = 0, \\ w_2 &= w_1(1 - h w_1) = 0. \end{aligned}$$

[2] Note that the exact error is

$$\int_0^1 \ln(x)\,\mathrm{d}x - Q_{[0,1]} \ln \approx -1 - (-0.6931471805599453) \approx -0.3068528194400547.$$

b) Formuler den implisitte Eulermetoden (med tilfeldig $h_i = t_{i+1} - t_i > 0$) for problemet. Vis at andregradslikningen som framkommer i metoden har to reelle røtter for approksimasjonen $w_{i+1} \approx y(t_{i+1})$ gitt av $w_i \approx y(t_i)$ og h_i, gitt at h_i er "liten nok". Finn de eksplisitte utrykkene for røttene og forklar hvilken av dem som bør velges i metoden.

Solution: Implicit Eulers method for this IVP is

$$w_{i+1} = w_i - h_i w_{i+1}^2.$$

$w_0 = y(0) = 1$. Thus at every iteration we need to sovle the quadratic equation

$$h_i w_{i+1}^2 + w_{i+1} - w_i = 0.$$

We solve this equation:

$$D = 1 + 4h_i w_i,$$
$$w_{i+1}^{\pm} = \frac{-1 \pm \sqrt{1 + 4h_i w_i}}{2h_i}.$$

In particular, if $w_i \geq 0$, or if $w_i < 0$ and $h_i < -1/(4w_i)$ then $D > 0$ and the quadratic equation has two real roots.

Which root should we choose? There are several ways in which one could argue here. For example, for small h we can use a first order Taylor series expansion $\sqrt{1 + 4h_i w_i} \approx 1 + 2h_i w_i$, and therefore $w_{i+1}^{\pm} \approx -1/(2h_i) \pm (1/(2h_i) + w_i)$. Since we expect that for small h we have $w_{i+1} \approx w_i$, we should select $w_{i+1} \approx w_{i+1}^{+}$ based on this information.

Another way of making this decision is to require that w_{i+1} behaves as much as the solution $y(t)$; for example, that it is positive. Indeed we know that $w_0 > 0$, and if $w_i > 0$ then $\sqrt{1 + 4h_i w_i} > 1$ and as a result $w_{i+1}^{-} < 0 < w_{i+1}^{+}$. Therefore we should choose w_{i+1}^{+}.

Oppgave 5

a) Beregn den diskrete Fouriertransformasjonen av $x = [1, 2, 3]^{\mathrm{T}}$.

Solution: direct computation.

$$y_0 = \frac{1}{\sqrt{3}} \sum_{j=0}^{3-1} x_j \exp\{-i2\pi j0/3\} = \frac{6}{\sqrt{3}} \approx 3.4641$$

$$
\begin{aligned}
y_1 &= \frac{1}{\sqrt{3}} \sum_{j=0}^{3-1} x_j \exp\{-i2\pi j1/3\} = \frac{1}{\sqrt{3}}[1\exp\{0\} + 2\exp\{-i2\pi/3\} + 3\exp\{-i4\pi/3\}] \\
&= \frac{1}{\sqrt{3}}[1 + 2\{-1/2 - i\sqrt{3}/2\} + 3\{-1/2 + i\sqrt{3}/2\}] = \frac{1}{\sqrt{3}}[-3/2 + i\sqrt{3}/2] \\
&= -\sqrt{3}/2 + i/2.
\end{aligned}
$$

Because $x \in \mathbb{R}^3$ we have $y_3 = \bar{y}_2 = -\sqrt{3}/2 - i/2$.

b) La $y = [y_0, y_1, \ldots, y_{n-1}]^{\mathrm{T}} \in \mathbb{C}^n$ være den diskrete Fouriertransformasjonen av vektoren $x = [x_0, x_1, \ldots, x_{n-1}]^{\mathrm{T}} \in \mathbb{C}^n$.

Nå konstruerer vi vektoren $\hat{x} = [x_0, x_{n-1}, x_{n-2}, \ldots, x_1]^{\mathrm{T}}$. Vis at den har en diskret Fouriertransformasjon gitt som $\hat{y} = [y_0, y_{n-1}, y_{n-2}, \ldots, y_1]^{\mathrm{T}}$.

Solution: Per definition of DFT:

$$
\begin{aligned}
\hat{y}_k &= \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \hat{x}_j \exp\{-i2\pi jk/n\} \\
&= \frac{1}{\sqrt{n}} \Big[\hat{x}_0 + \hat{x}_1 \exp\{-i2\pi 1k/n\} + \cdots + \hat{x}_{n-1} \exp\{-i2\pi(n-1)k/n\}\Big] \\
&= \frac{1}{\sqrt{n}} \Big[x_0 + x_{n-1} \exp\{-i2\pi 1k/n\} + \cdots + x_1 \exp\{-i2\pi(n-1)k/n\}\Big] \\
&= \frac{1}{\sqrt{n}} \Big[x_0 + x_1 \exp\{-i2\pi(n-1)k/n\} + \cdots + x_{n-1} \exp\{-i2\pi 1k/n\}\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
y_{n-k} &= \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j \exp\{-i2\pi j(n-k)/n\} \\
&= \frac{1}{\sqrt{n}} \Big[x_0 + x_1 \exp\{-i2\pi 1(n-k)/n\} + \cdots + x_{n-1} \exp\{-i2\pi(n-1)(n-k)/n\}\Big].
\end{aligned}
$$

Direct comparison shows that $\hat{y}_0 = y_0$ (then all $\exp\{\cdot\} = \exp\{0\} = 1$). To conclude the proof we need to show that the coefficients in front of x_j in two formulas agree. Indeed, the coefficient in front of x_j in the formula for $\hat{y}_k$ is

$$
\exp\{-i2\pi(n-j)k/n\} = \underbrace{\exp\{-i2\pi k\}}_{=1} \exp\{i2\pi jk/n\} = \exp\{i2\pi jk/n\},
$$

and similarly, the coefficient in front of x_j in the formula for y_{n-k} is:

$$\exp\{-i2\pi j(n-k)/n\} = \underbrace{\exp\{-i2\pi j\}}_{=1} \exp\{i2\pi jk/n\} = \exp\{i2\pi jk/n\},$$

where we used the face that $\exp\{-i2\pi j\} = \exp\{-i2\pi k\} = 1$ for all integers j, k. Thus the proof is concluded.

c) La $n = 2^p$, $p \in \mathbb{N}$, og $t_j = c + j(d-c)/n$, $j = 0, \dots, n-1$ være en samling av uniformt distribuerte punkter på intervallet $[c, d]$. Vi vil finne en kurve som passerer gjennom(interpolerer) punktene i datasettet $(t_0, x_0), \dots, (t_{n-1}, x_{n-1})$.

I dette kurset har vi sett på to mulige metoder for å gjøre dette: polynominterpolasjon, (her vist i Newtons form)

$$P(t) = f[t_0] + f[t_0, t_1](t - t_0) + \dots + f[t_0, \dots, t_{n-1}](t - t_0) \cdots (t - t_{n-2}),$$

og trigonometrisk interpolasjon:

$$Q(t_j) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} y_k \exp\{i2\pi kj/n\} = \sum_{k=0}^{n-1} y_k \exp\left\{\frac{i2\pi k(t_j - c)}{d - c}\right\} / \sqrt{n}.$$

Gi et overslag på antallet av elementære operasjoner[3] som trengs til å beregne:

- *alle* Newtons differanser[4] $f[t_0], f[t_0, t_1], \dots, f[t_0, \dots, t_{n-1}]$;
- *alle* trigonometriske interpolasjonskoeffisientene $y_0, \dots, y_{n-1}$ ved hjælp av FFT algoritmen.

Sammenlign to vurderingene og bestem, hva er raskest for store n.

Solution: The coefficients $y_0, \dots, y_{n-1}$ are efficiently computed using FFT, which requires $O(n \log n)$ operations.

Computing each Newton's divided difference requires one division and two subtractions. There are $n-1$ divided differences $f[t_0, t_1], \dots, f[t_{n-2}, t_{n-1}]$; $n-2$ divided differences $f[t_0, t_1, t_2], \dots, f[t_{n-3}, t_{n-2}, t_{n-1}]$, ..., and finally 1 difference $f[t_0, t_1, \dots, t_{n-1}]$. Therefore, computing all Newton's divided differences requires $(n-1) + (n-2) + \dots + 1 = O(n^2)$ subtractions and divisions.

For large n we have $n > \log n$ and therefore computing the trigonometric expansion coefficients is faster.

[3]Elementære operasjoner ≈ addisjon, subtraksjon, multiplikasjon, divisjon

[4]Newton's divided differences