Norges teknisk–naturvitenskapelige universitet
Institutt for Matematiske Fag

TMA4329 Intro til vitensk. beregn. V2016

ving 6

[S]=T. Sauer, Numerical Analysis, Second International Edition, Pearson, 2014

"Teorioppgaver"

1 Oppgave 10.1.1

Solution:

(a) $x = [0, 1, 0, -1]$,

$$y_k = \frac{1}{2}\sum_{j=0}^{3} x_j \exp(-i2\pi jk/4) = \frac{1}{2}[\exp(-i2\pi k/4) - \exp(-i6\pi k/4)]$$

Thus $y_0 = 0$, $y_1 = [\exp(-i\pi/2) - \exp(-i3\pi/2)]/2 = [-i-i]/2 = -i$, $y_2 = [\exp(-i\pi) - \exp(-i3\pi)]/2 = [-1-(-1)]/2 = 0$, $y_3 = [\exp(-i3\pi/2) - \exp(-i9\pi/2)]/2 = [i-(-i)]/2 = i$,

(b) $x = [1, 1, 1, 1]$, $y_0 = 2$, $y_1 = [\exp(-i\pi/2)+\exp(-i\pi)+\exp(-3i\pi/2)+\exp(-2i\pi)]/2 = [-i-1+i+1]/2 = 0$, $y_2 = 0$, $y_3 = 0$.

(c) $x = [0, -1, 0, 1]$, which is just negative of x in (a). Since DFT is linear $\implies$ $y = [0, i, 0, -i]$.

(d) $x = [0, 1, 0, -1, 0, 1, 0, -1]$, that is, the repetition of x in (a) twice. Thus

$$\begin{aligned}
y_k &= \frac{1}{\sqrt{8}}\sum_{j=0}^{7} x_j \exp(-i2\pi jk/8) \\
&= \frac{1}{2\sqrt{2}}\sum_{j=0}^{3} x_j \exp(-i2\pi jk/8) + \frac{1}{2\sqrt{2}}\sum_{j=4}^{7} x_j \exp(-i2\pi jk/8) \\
&= \frac{1}{2\sqrt{2}}\sum_{j=0}^{3} x_j \exp(-i2\pi jk/8) + \frac{1}{2\sqrt{2}}\exp(-i2\pi 4k/8)\sum_{j=0}^{3} x_j \exp(-i2\pi jk/8) \\
&= \frac{1}{2\sqrt{2}}[1+\exp(-i\pi k)]\sum_{j=0}^{3} x_j \exp(-i2\pi jk/8).
\end{aligned}$$

In particular, the factor in the square brackets equals 2 for even k and 0 for odd k, and therefore $y_k = 0$ for all odd $k = 1, 3, 5, 7$. Furthermore, for even $k = 0, 2, 4, 6$ the sum in the last expression equals that of the sum in (a) for $k/2$. Thus $y_0 = 0$, $y_2 = -i\sqrt{2}$, $y_4 = 0$, $y_6 = i\sqrt{2}$.

2 Oppgave 10.1.7

3 Prove the following identity (Parseval's theorem for DFT): Let $y^{(1)} = \mathrm{DFT}(x^{(1)}$ and $y^{(2)} = \mathrm{DFT}(x^{(2)})$. Then $\sum_{j=0}^{n-1} x_j^{(1)} \bar{x}_j^{(2)} = \sum_{j=0}^{n-1} y_k^{(1)} \bar{y}_k^{(2)}$.
Hint: $y^{(1,2)} = F_n x^{(1,2)}$.

Solution:

$$\sum_{j=0}^{n-1} y_k^{(1)} \bar{y}_k^{(2)} = [\bar{y}^{(2)}]^T y^{(1)} = [\bar{F}_n \bar{x}^{(2)}]^T F_n x^{(1)} = [\bar{x}^{(2)}]^T \underbrace{\bar{F}_n^T F_n}_{=I} x^{(1)} = \sum_{j=0}^{n-1} x_j^{(1)} \bar{x}_j^{(2)},$$

since $\bar{F}_n^T = F_n^{-1}$ (see for example (10.11) in [S]).

4 Prove the following identity (circular shift theorem for DFT): Let $\tilde{x}_j = \exp(i2\pi jm/n)$, where $i^2 = -1$ and m is an integer. Let $y = \mathrm{DFT}(x)$, and $\tilde{y} = \mathrm{DFT}(\tilde{x})$. Show that $\tilde{y}_k = y_{k-m}$, where the subscript $k - m$ is understood modulo n (in other words, the sequence y is assumed to be periodically repeating with period n).

Solution:

$$\begin{aligned}\tilde{y}_k &= \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \tilde{x}_j \exp(-i2\pi jk/n) = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j \exp(i2\pi jm/n) \exp(-i2\pi jk/n) \\ &= \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j \exp(-i2\pi j(k-m)/n) = y_{k-m},\end{aligned}$$

since $k \mapsto \exp(-i2\pi jk/n) = \cos(2\pi jk/n) - i\sin(2\pi jk/n)$ is a periodic function with period n.

"Computeroppgaver"

5 Implement Cooley–Tukey's algorithm for computing DFT:

```
function [y,nop] = myfft(x)
% Implementation of FFT
%
% Input: x, supposed to be a vector of length 2^p
%
% Output: y:   FFT(x)
%         nop: number of operations required
%
```

Verify it against Matlab's FFT (recall that Matlab uses a different scaling of DFT). Check that the number of operations needed by Cooley–Tukey's algorithm scales as $O(N \log(N))$ by plotting the number of iterations vs. N and $N \log(N)$ on a log-log plot for a range of $N = 2^p$.

Solution:

See `myfft.m` and `check_myfft.m`.