“Teorioppgaver”

1. Oppgave 6.1.3 (b), (e), s. 291, [S]

Solution:
(b): Separation of variables leads to the equation \( \int \frac{dy}{y} = \int t^2 \, dt \), or \( \log(y) = \frac{t^3}{3} + C \) where \( C \) is an arbitrary integration constant. Thus \( y(t) = \exp(\frac{t^3}{3} + C) \), where \( C = 0 \) from the equation \( y(0) = 1 \).

(e): Here we get \( \int y^2 \, dy = \int 1 \, dt \), or \( y^3 / 3 = t + C \), \( y(t) = (3t + 3C)^{1/3} \). Finally \( y(0) = 1 \) and therefore \( 3C = 1 \).

2. Oppgave 6.1.4 (a), (b), s. 291, [S]

Solution:
(a): The general solution to the homogeneous system \( y' = y \) is \( y_0(t) = C \exp(t) \), where \( C \) is an arbitrary constant.

Further, a particular solution to the system \( y' = t + y \) can be search in the form \( y_1(t) = at + b \), from which it follows that \( a = b = -1 \).

Thus we can put \( y(t) = y_0(t) + y_1(t) \). The constant \( C \) is then determined from the equation \( y(0) = 0 \) and thus \( y(t) = -t - 1 + \exp(t) \).

(b): Similarly to (a), \( y_0(t) = C \exp(-t) \), \( y_1(t) = t - 1 \), and as a result \( y(t) = t - 1 + \exp(-t) \).

3. Oppgave 6.1.9, s. 292, [S]

Solution:
(a): Here \( f(t, y) = t \) - independent from \( y \) and thus \( f \) is uniformly Lipschitz continuous with respect to \( y \) on \( [a, b] \times (-\infty, +\infty) \) with the constant \( L = 0 \). Theorem 6.2 then guarantees the existence and uniqueness of solutions on an arbitrary interval.

(b), (c): Here \( f(t, y) = \pm y \), which is a linear function with the slope \( \pm1 \). In either case \( f \) is uniformly Lipschitz continuous with respect to \( y \) with constant \( L = 1 \) on
\([a, b] \times (-\infty, +\infty)\), regardless of \(a, b\). Theorem 6.2 then guarantees the existence and uniqueness of solutions on an arbitrary interval.

**(d):** Here \(f(t, y) = -y^3\). This function is independent from \(t\) and is continuously differentiable with respect to \(y\). Thus it is uniformly Lipschitz continuous on any finite square \(S = [a, b] \times [\alpha, \beta]\) with the Lipschitz constant \(L = \max_{y \in [\alpha, \beta]} |df/dy| = \max_{y \in [\alpha, \beta]} |3y^2|\). Note that the Lipschitz constant depends and “grows” with the interval \([\alpha, \beta]\).

Thus Theorem 6.2 guarantees the existence and uniqueness of solutions on some sub-interval \([a, c]\), \(a < c \leq b\). As a result we cannot guarantee existence of solutions on the whole interval \([0, 1]\), only on a sub-interval.

4 Oppgave 6.1.10, s. 292, [S]

Solution:

\[\text{(a)}\]

\[\text{(b)}\]

\[\text{(c)}\]

\[\text{(d)}\]

5 Oppgave 6.1.11, s. 292, [S]

Solution:

**(a):** \(y(t) = t^2/2 + y(0)\). Theorem 6.3 is clearly verified with \(L = 0\).

**(b,c):** \(y(t) = y(0) \exp(\pm t)\). Theorem 6.3 is verified with \(L = 1\). In fact, in (c) any non-negative \(L\) or even \(L \geq -1\) is sufficient, but this is of course is not a valid value for the Lipschitz constant.
(d): If \( y(0) = 0 \) we can use the solution \( y(t) \equiv 0 \). For \( y(0) = 1 \) we can use separation of variables to find that \( y(t) = (2t + C)^{-1/2} \), where \( 1 = y(0) = C^{-1/2} \). Note that the difference between the two solutions decreases with time, and therefore the estimate of Theorem 6.3 holds with any \( L \geq 0 \) in this case.

6 Oppgave 6.2.2, s. 302, [S]

Solution:
(a):

\[
\begin{array}{ccc}
t_i & y_i &
\end{array}
\]

\[
\begin{array}{ccc}
0.000000e+00 & 0.000000e+00
2.500000e-01 & 3.125000e-02
5.000000e-01 & 1.416016e-01
7.500000e-01 & 3.533020e-01
1.000000e+00 & 6.948557e-01
\end{array}
\]

\[
\text{err} = 0.0234
\]

(b):

\[
\begin{array}{ccc}
t_i & y_i &
\end{array}
\]

\[
\begin{array}{ccc}
0.000000e+00 & 0.000000e+00
2.500000e-01 & 3.125000e-02
5.000000e-01 & 1.103516e-01
7.500000e-01 & 2.268372e-01
1.000000e+00 & 3.725290e-01
\end{array}
\]

\[
\text{err} = 0.0046
\]

7 Oppgave 6.3.3, s. 302, [S]

Solution:

We introduce a new variable \( z = y' \) so that \( z' = y'' \).

(a):

\[
\begin{pmatrix}
y' \\
z
\end{pmatrix} = \begin{pmatrix}
z \\
y
\end{pmatrix}
\]

(b):

\[
\begin{pmatrix}
y' \\
z
\end{pmatrix} = \begin{pmatrix}
z \\
2tz - 2y
\end{pmatrix}
\]
(c):

\[
\begin{pmatrix}
  y' \\
  z
\end{pmatrix} = \begin{pmatrix}
  z \\
  tz + y
\end{pmatrix}
\]

Consider the initial value problem

\[
y'(t) = \lambda y(t), \quad t > 0
\]
\[
y(0) = y_0,
\]

where \( \lambda \in \mathbb{C} \). Its solution is \( y(t) = y_0 \exp(\lambda t) \).

Suppose that we use a numerical method (such as e.g. forward Euler or explicit trapezoid) to solve this problem starting from a point \( w_0 = y_0 \). The stability region for the method is a set of points \( z = \lambda h \) in the complex plane, such that the numerical solution \( (w_0, w_1, \ldots) \) stays bounded (i.e., \( \exists C > 0 : \forall i, |w_i| \leq C \)).

Find the stability region for (a) forward Euler method; (b) explicit Trapezoid method.

**Solution:**

(a): In this case \( w_k = w_{k-1} + hf(t_{k-1}, w_{k-1}) = (1 + h\lambda)w_{k-1} = (1 + h\lambda)^k w_0 \). Thus \( w_k \) stays bounded iff \( |1 + h\lambda| \leq 1 \). That is, the stability region for the forward Euler method is a circle in the complex plane of radius 1 around the point \(-1\).

Note: this implies, in particular, that if \( \lambda = i\omega \) is purely imaginary then there is no \( h > 0 \) such that \( h\lambda \) is in the stability region. Thus whereas \( \exp(i\omega t) = \cos(\omega t) + i\sin(\omega t) \) is oscillatory (bounded) in this case, the Euler’s method will result in an unbounded solution (long term behaviour) regardless of how small we chose \( h \).

(b): Now we have \( w_k = w_{k-1} + h/2[f(t_{k-1}, w_{k-1}) + f(t_{k-1} + h, w_{k-1} + hf(t_{k-1}, w_{k-1}))] + w_{k-1} + h/2[\lambda w_{k-1} + \lambda(w_{k-1} + h\lambda w_{k-1})][1 + h\lambda + 0.5(h\lambda)^2]w_{k-1} = [1 + h\lambda + 0.5(h\lambda)^2]^k w_0 \). Thus \( z = h\lambda \) is in the stability region of the explicit trapezoid method if and only if \( |1 + z + 0.5z^2| \leq 1 \).

For an arbitrary purely imaginary number \( \lambda = i\omega \) and any \( h > 0 \) we have \( |p(i\omega h)| = |1 - 0.5\omega^2 h^2 + i\omega h| = |(1 - 0.5\omega^2 h^2)^2 + \omega^2 h^2|^{1/2} = |1 + \omega^4 h^4/4|^{1/2} > 1 \) as in the case of forward Euler, with the same implications. However, for small \( h > 0 \) we can use a first order Taylor series expansion to get \( [1 + \omega^4 h^4/4]^{1/2} \approx 1 + \omega^4 h^4/8 \) which is much closer to 1 than \( |1 + i\omega h| = (1 + \omega^2 h^2)^{1/2} \approx 1 + \omega^2 h^2/2 \) (forward Euler).

Here is the plot of stability regions:

![Stability Regions](image-url)
“Computeroppgaver”

9 Oppgave 6.1.5, s. 293, [S]

Solution:
See oppgave_6_1_5.m

Repeat the previous exercise, but use the explicit trapezoid method instead.

Solution:
See oppgave_10.m

11 Oppgave 6.3.10, s. 314, [S]. Use the initial conditions specified in the book but different masses: $m_1 = m_3 = 0.03$, $m_2 = 0.3$. Use the explicit Trapezoid method.

Solution:
See three_body_problem.m
Oppgave 6.3.11, s. 314, [S]. Use the explicit Trapezoid method.

Solution:
See three_body_problem.m