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TMA4305 Partial Differential Equations
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Problem 1 Find a solution u to $u_t + \frac{1}{3}(u^3)_x = 0$, valid for all $x \in \mathbb{R}$ and small positive t, with initial data

$$u(0,x) = \begin{cases} -1 & x < 0, \\ x-1 & 0 \le x \le 1, \\ 0 & x > 1. \end{cases}$$

At what time does a classic solution cease to exist? What happens next?

Problem 2 Assume that $u \in C\big([0,T]\times\overline{\Omega}\big) \cap C^2\big((0,T)\times\Omega\big)$ satisfies

$$\begin{aligned} u_t - \Delta u &= u - u^3 && \text{in } (0,T)\times\Omega, \\ |u| &\le 1 && \text{in } \big(\{0\}\times\Omega\big) \cup \big([0,T]\times\partial\Omega\big), \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded region. Show that then $|u| \le 1$ in $[0,T]\times\overline{\Omega}$.

Problem 3 Show that if u is a positive, harmonic function on a region $\Omega \subseteq \mathbb{R}^n$, then

$$|\nabla u(\boldsymbol{x})| \le \frac{n\,u(x)}{\operatorname{dist}(\boldsymbol{x},\partial\Omega)}$$

for all $\boldsymbol{x} \in \Omega$, where $\operatorname{dist}(\boldsymbol{x},\partial\Omega) = \inf\big\{\,|\boldsymbol{y}-\boldsymbol{x}| \,\big|\, \boldsymbol{y} \in \partial\Omega\,\big\}$.

If $\Omega = \mathbb{R}^n$, we define $\operatorname{dist}(\boldsymbol{x},\partial\Omega) = \infty$. Show that u is indeed constant in this case.

Hint: There are at least three approaches that work: Use Harnack's inequality, or Poisson's integral formula, or (perhaps easiest) adapt the proof of Proposition 6 in the *Harmonicfunctionology* note.

Problem 4 Assume that $u \in C(\mathbb{R}^n)$ is harmonic in the upper halfspace $\big\{\boldsymbol{x} \in \mathbb{R}^n \,\big|\, x_n > 0\big\}$, and also antisymmetric in x_n; i.e.,

$$u\big(\sigma(\boldsymbol{x})\big) = -u(\boldsymbol{x}), \quad \text{where } \sigma(x_1,\dots,x_{n-1},x_n) = (x_1,\dots,x_{n-1},-x_n).$$

Show that u is harmonic in $\mathbb{R}^n$.

Hint: It is trivial to show that u is also harmonic in the lower half space $x_n < 0$. The trouble is at $x_n = 0$. Given a real number $r > 0$, consider the ball B of radius r, centred at $\mathbf{0}$. Construct a suitable harmonic function on B and show that it coincides with u in B. For that, you will need to also consider the "upper half ball" $\big\{\boldsymbol{x} \in B \,\big|\, x_n > 0\big\}$. Use the antisymmetry for all it's worth.

Problem 5 *In this problem, subscripts on functions do* not *indicate partial derivatives. Instead, for a function of four variables $\psi(t,\boldsymbol{x})$ with $t \in \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{R}^3$, we write $\psi_t(\boldsymbol{x}) = \psi(t,\boldsymbol{x})$. Think of it as a function on $\mathbb{R}^3$, varying with time. We use ∂_t as a shorthand notation for $\partial/\partial t$.*

Also, we use the shorthand notation $\mathcal{D}(\mathbb{R}^3) = C_c^\infty(\mathbb{R}^3)$ (the textbook uses $C_0^\infty(\mathbb{R}^3)$).

Given a real number $t > 0$, define the t-ball $B_t \subset \mathbb{R}^3$ and the t-sphere $S_t \subset \mathbb{R}^3$ by

$$B_t = \{\boldsymbol{x} \in \mathbb{R}^3 \mid |\boldsymbol{x}| < t\}, \quad S_t = \partial B_t = \{\boldsymbol{x} \in \mathbb{R}^3 \mid |\boldsymbol{x}| = t\}.$$

Further, define distributions $b_t, s_t \in \mathcal{D}'(\mathbb{R}^3)$ by

$$(b_t,\psi) = \int_{B_t} \psi \, \mathrm{d}^3\boldsymbol{x}, \quad (s_t,\psi) = \int_{S_t} \psi \, \mathrm{d}S \qquad \text{for } \psi \in \mathcal{D}(\mathbb{R}^3).$$

a. Explain why the above definitions really do specify distributions b_t and s_t. Show that

$$\lim_{t\to 0} \frac{s_t}{4\pi t^2} = \delta \quad \text{and} \quad (b_t,\psi) = \int_0^t (s_\tau,\psi)\,\mathrm{d}\tau.$$

b. Show that $\partial_t s_t = 2t^{-1}s_t + \Delta b_t$.

Use the straightforward definition: That $\partial_t s_t$ is a distribution satisfying $\dfrac{\mathrm{d}}{\mathrm{d}t}(s_t,\psi) = (\partial_t s_t,\psi)$ for any $\psi \in \mathcal{D}(\mathbb{R}^3)$.

Hint: Start with $(s_t,\psi) = t^2 \int_{S_1} \psi(t\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x})$, take the derivative, move one integral back to S_t, and use the divergence theorem.

c. Show that $\partial_t \Delta b_t = \Delta \partial_t b_t = \Delta s_t$.

Let $w_t = \dfrac{s_t}{4\pi t}$, calculate $\partial_t w_t$, and show that $\partial_{tt} w_t = \Delta w_t$ for $t > 0$.

d. Define $w \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^3)$ by

$$(w,\psi) = \int_0^\infty (w_t,\psi_t)\,\mathrm{d}t \qquad \text{for } \psi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^3).$$

Show that w is a fundamental solution to the wave operator $\partial_{tt} - \Delta$.

Hint: Start with $(\partial_{tt} w,\psi) = (w,\partial_{tt}\psi)$, use the definition of w, and proceed to move the t derivatives back onto w_t. You may freely use the "product rule" (Leibniz' formula)

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_t,\psi_t) = (\partial_t u_t,\psi_t) + (u_t,\partial_t \psi_t)$$

where u_t is any of the time dependent distributions encountered, so long as $\partial_t u_t$ exists. Note that partial integration is nothing but Leibniz' formula "in reverse".