TMA4305 Partial Differential Equations

6 August 2020

Solution outline

PRELIMINARY VERSION may contain mistakes

Problem 1 Looking at $u_t + f(u)_x = 0$ $\quad f(u) = \arctan u$

The characteristic speed is $\quad f'(u) = \dfrac{1}{\sqrt{1+u^2}}$

The characteristic starting at $x = \xi$ has speed

$v(\xi) = f'(u(\xi)) = f'(\xi) = 1/\sqrt{1+\xi^2}$

and equation $\quad x = \xi + \dfrac{t}{\sqrt{1+\xi^2}}$.

Since $u = u(0,\xi) = \xi$ on this characteristic, we have

$$x = u + \frac{t}{1+u^2}$$

which defines u implicitly from (t,x).

The equation can be solved for u so long as $\dfrac{\partial x}{\partial u} > 0$.

We find $\dfrac{\partial x}{\partial u} = 1 - \dfrac{2tu}{(1+u^2)^2}$, so we need $\dfrac{2u}{(1+u^2)^2} < \dfrac{1}{t}$

for this to hold. Look for the maximum value of this.

$$\frac{d}{du} 2u(1+u^2)^{-2} = 2(1+u^2)^{-2} - 8u^2(1+u^2)^{-3}$$
$$= 2(1+u^2)^{-3}(1+u^2-4u^2)$$
$$= 2(1+u^2)^{-3}(1-3u^2)$$

So $\dfrac{2u}{(1+u^2)^2}$ achieves its maximum at $u = \sqrt{1/3}$

and we find $\left.\dfrac{2u}{(1+u^2)^2}\right|_{u=\sqrt{1/3}} = \dfrac{2\sqrt{1/3}}{(4/3)^2} = \dfrac{3}{8}\sqrt{3}$

and the classical solution is valid for

$$t < T = \frac{1}{\frac{3}{8}\sqrt{3}} = \frac{8}{9}\sqrt{3}.$$

The characteristic for $u = \sqrt{1/3}$ starts at $x = \sqrt{1/3}$

and has speed $f'(\sqrt{1/3}) = \dfrac{1}{\sqrt{1+(\sqrt{1/3})^2}} = \dfrac{3}{4}.$

Nearby characteristics start colliding at $t = \frac{8}{9}\sqrt{3}$,

at position $x = \sqrt{1/3} + \frac{3}{4}\cdot\frac{8}{9}\sqrt{3} = \frac{1}{3}\sqrt{3} + \frac{2}{3}\sqrt{3} = \sqrt{3}$

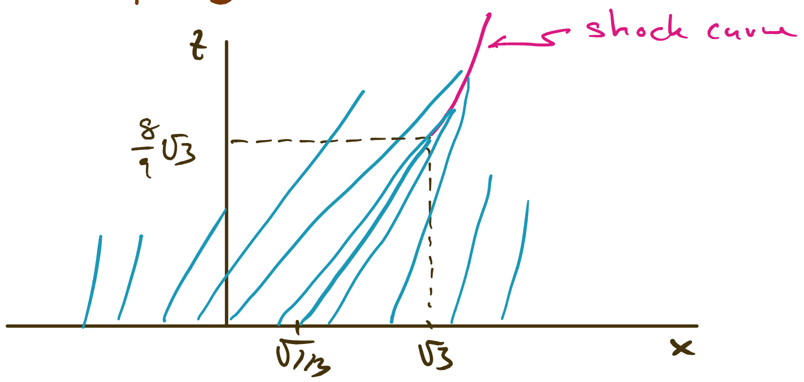
The resulting shock will have a speed given by

$$s = \frac{f(u_r) - f(u_\ell)}{u_r - u_\ell} \approx f'(u)$$

with $u = \sqrt{1/3}$, since u_r and u_ℓ will be close to this value. Thus the initial shock speed is

$$s \approx f'(\sqrt{1/3}) = \frac{3}{4}$$

with equality in the limit as $t \searrow T$.



(I don't know if it speeds up or slows down... ☺)

Problem 2

The smallest eigenvalue λ_1 is the minimum of the Rayleigh quotient $\frac{\|\nabla u\|^2}{\|u\|^2}$ for $u \in H_0^1(\Omega)$.

Thus $\lambda_1 \leq \frac{\|\nabla u\|^2}{\|u\|^2}$ for any $u \in H_0^1(\Omega)$.

Picking $u(x,y) = (1-r)\,y = (r-r^2)\sin\theta$

we note that $u=0$ on $\partial\Omega$, and indeed $u \in H_0^1(\Omega)$.

We find

$$\|u\|^2 = \int_\Omega |u|^2\,dx\,dy = \int_0^1 \int_0^\pi (r-r^2)^2 \sin^2\theta \, d\theta \, r\,dr$$

$$= \int_0^1 (r^3 - 2r^4 + r^5)\,dr \int_0^\pi \sin^2\theta\,d\theta$$

$$= \left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right)\cdot\frac{\pi}{2} = \frac{15-24+10}{60}\cdot\frac{\pi}{2} = \frac{\pi}{120}$$

$$|\nabla u|^2 = \left|\frac{\partial u}{\partial r}\right|^2 + \left|\frac{1}{r}\frac{\partial u}{\partial \theta}\right|^2 = \big((1-2r)\sin\theta\big)^2 + \big((1-r)\cos\theta\big)^2$$

and a similar calculation to the above yields

$$\|\nabla u\|^2 = \int_\Omega |\nabla u|^2\,dx\,dy = \int_0^1 (1-2r)^2\,r\,dr\cdot\frac{\pi}{2} + \int_0^1 (1-r)^2\,r\,dr\cdot\frac{\pi}{2}$$

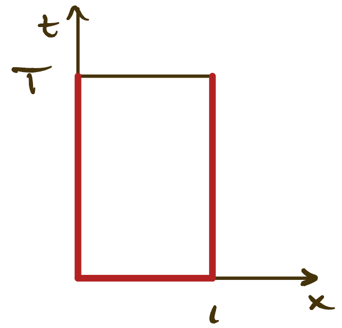
$$= \int_0^1 \left(1-4r+4r^2+1-2r+r^2\right) r\,dr\cdot\frac{\pi}{2}$$

$$= \int_0^1 (2r - 6r^2 + 5r^3)\,dr\cdot\frac{\pi}{2}$$

$$= \left(1 - 2 + \frac{5}{4}\right)\cdot\frac{\pi}{2} = \frac{\pi}{8}$$

So $\frac{\|\nabla u\|^2}{\|u\|^2} = \frac{120}{8} = 15$, and so $\lambda_1 \leq 15$.

Problem 3



The parabolic boundary (red in the picture) is

$$\underbrace{\{0\}\times[0,1]}_{\text{initial boundary}} \cup \underbrace{[0,T]\times\{0,1\}}_{\text{spatial boundary}}$$

When we subtract the parabolic boundary, we are left with $(0,T]\times(0,1)$ i.e., $0<t\le T$, $0<x<1$.
If a smooth function u has a local maximum at such a point, then $u_x=0$, $u_{xx}\le 0$, and $u_t\ge 0$ ($u_t=0$ if $t<T$; but at $t=T$, only $u_t\ge 0$ remains).
These plugged into the equation $u_t=u_{xx}+\sin(u_x)-\varepsilon$ yield $0\le u_t\le -\varepsilon$, which is a contradiction. So u has no local maximum except at the parabolic boundary.

Write Γ for the parabolic boundary.
If $u_t=u_{xx}+\sin(u_x)$, let $v(t,x)=u(t,x)-t\varepsilon$.
Then $v_x=u_x$, $v_{xx}=u_{xx}$, and
$v_t=u_t-\varepsilon=u_{xx}+\sin(u_x)-\varepsilon=v_{xx}+\sin(v_x)-\varepsilon$.
By the first part, v has no (local) maximum on $\bar\Omega\setminus\Gamma$.
But v is continuous on the compact set $\bar\Omega$, so it has a maximum. Therefore $v\le\max_\Gamma v$. So
$u=v+t\varepsilon\le\max_\Gamma v+T\varepsilon\le\max_\Gamma u+T\varepsilon$. Now let $\varepsilon\searrow 0$.

Problem 4 We have a linear equation, with initial and boundary conditions also linear. So to show uniqueness, it is enough to put $f=0$ and $g=h=0$ and to prove that $u=0$.

The equation yields $u_{tt}u_t + u_{xxxx}u_t = 0$, hence

$$\frac{d}{dt}\int_0^1 \tfrac{1}{2}u_t^2\,dx + \int_0^1 u_{xxxx}u_t\,dx = 0.$$

Now

$$\int_0^1 u_{xxxx}u_t\,dx = \underbrace{\left[u_{xxx}u_t\right]_{x=0}^{x=1}}_{\text{zero, see (A)}} - \int_0^1 u_{xxx}u_{xt}\,dx$$

$$= -\underbrace{\left[u_{xx}u_{xt}\right]_{x=0}^{1}}_{\text{zero, see (B)}} + \int_0^1 u_{xx}u_{xxt}\,dx$$

$$= \int_0^1 \tfrac{1}{2}(u_{xx}^2)_t\,dx = \frac{d}{dt}\int_0^1 \tfrac{1}{2}u_{xx}^2\,dx,$$

so $\displaystyle\frac{d}{dt}\int_0^1\left(\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_{xx}^2\right)dx = 0.$

Since the integral is 0 for $t=0$, it is zero for $t>0$.
In particular $u_t=0$, so $u(t,x)=u(0,x)=0$.

(A) Because $u(t,x)=0$ for $x=0$ and $x=1$, all $t>0$

(B) Because $u_x(t,0)=0$ for $t>0$, so $u_{xt}(t,0)=0$ and because $u_{xx}(t,1)=0$.

Problem 5 We have an explicit solution for the wave equation with initial data $u(0,\vec{x}) = g(\vec{x})$ and $u_t(0,\vec{x}) = h(\vec{x})$. In this case, $g(\vec{x})=0$, so we only need half of the formula:

$$u(t,\vec{x}) = t \fint_{\partial B(\vec{x};t)} h(y)\, dS(y) = t \fint_{\partial B(\vec{x};t)} \frac{dS(\vec{y})}{\sqrt{1+|\vec{y}|^2}}$$

$$= t \fint_{S^2} \frac{dS(\vec{z})}{\sqrt{1+|\vec{x}+t\vec{z}|^2}}$$

$$= \fint_{S^2} \frac{dS(\vec{z})}{\sqrt{\frac{1}{t^2} + \left|\frac{\vec{x}}{t}+\vec{z}\right|^2}} \longrightarrow 1 \quad \text{when } t\to\infty$$

Notes: $\fint \cdots$ is the average integral.

$B(\vec{x};t)$ is the ball of radius t centred at $\vec{x}$,

$\partial B(\vec{x};t)$ is its boundary (a sphere)

and $S^2 = \partial B(\vec{0};1)$ (the unit sphere).

The final limit is justified by noting that $|\vec{z}|=1$ and $\vec{x}$ is fixed, so the integrand converges uniformly to 1 when $t\to\infty$.