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TMA4305 Partial Differential Equations 6 August 2020

Problem 1 Consider the initial value problem

$$u_t + (\arctan u)_x = 0,$$
$$u(0,x) = x.$$

Show that the problem has a smooth solution valid for all $x \in \mathbb{R}$ and all sufficiently small $t \geq 0$. Derive an implicit equation that determines this solution. At what time will a shock start to form? What is the initial location and speed of this shock? Draw a picture showing the characteristics and the beginning of the shock curve.

Problem 2 Determine an upper bound for the smallest eigenvalue of the Laplace operator on the semidisk $\{\,(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1,\ y > 0\,\}$. Hint: Try $(1-r)y$ and use polar coordinates.

Problem 3 Assume that $u \in C^2([0,T]\times[0,1])$ solves the equation

$$u_t = u_{xx} + \sin(u_x) - \varepsilon$$

where $\varepsilon > 0$. What is the parabolic boundary of $(0,T)\times(0,1)$? Show that u cannot have a local maximum at any point of $[0,T]\times[0,1]$ *except* at the parabolic boundary.

Next, assume that u satisfies instead

$$u_t = u_{xx} + \sin(u_x).$$

Show that u achieves its maximum on the parabolic boundary. (Hint: $u - \varepsilon t$.)

Problem 4 Consider the initial–boundary value problem

$$\begin{aligned} u_{tt} + u_{xxxx} &= f(t,x) && \text{for } x \in (0,1) \text{ and } t > 0,\\ u(0,x) &= g(x) && \text{for } x \in (0,1),\\ u_t(0,x) &= h(x) && \text{for } x \in (0,1),\\ u(t,0) = u_x(t,0) &= 0 && \text{for } t > 0,\\ u(t,1) = u_{xx}(t,1) &= 0 && \text{for } t > 0. \end{aligned}$$

where f, g and h are given functions.

Use an energy functional to show that any classical solution is unique, if it exists. Hint: Start by multiplying the equation with u_t, integrate over x, and use partial integration.

Problem 5 Assume that $u\colon [0,\infty)\times\mathbb{R}^3 \to \mathbb{R}$ solves the wave equation with initial data:

$$\begin{aligned} u_{tt} - \Delta u &= 0 \qquad \text{for } t > 0, \\ u(0,\boldsymbol{x}) &= 0, \\ u_t(0,\boldsymbol{x}) &= \frac{1}{\sqrt{1+|\boldsymbol{x}|^2}}. \end{aligned}$$

Compute $\lim_{t\to\infty} u(t,\boldsymbol{x})$. Hint: Do not try to calculate $u(t,\boldsymbol{x})$.