TMA4305 Partial differential equations 2019-12-04

Solution

Problem 1

We find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2\,\mathrm{d}^n\boldsymbol{x} = 2\int_\Omega uu_t\,\mathrm{d}^n\boldsymbol{x} = 2\int_\Omega u\,\Delta u\,\mathrm{d}^n\boldsymbol{x}$$
$$= 2\int_\Omega \big(\nabla\cdot(u\nabla u) - |\nabla u|^2\big)\,\mathrm{d}^n\boldsymbol{x} < 2\int_{\partial\Omega}\boldsymbol{n}\cdot u\nabla u\,\mathrm{d}S = 2\int_{\partial\Omega} u\frac{\partial u}{\partial n}\,\mathrm{d}S.$$

For the first equality we need continuity of u_t, so we can take the derivative inside the integral. The second equality uses the heat equation, while the third equality is just the product rule, requiring u to have second order derivatives in the $\boldsymbol{x}$ variable. The inequality uses the divergence theorem on the first term (equality), requiring $u \in C^2$ and sufficient regularity of the boundary of Ω (piecewise C^1 is sufficient, but far from necessary). The final equality is just the definition of normal derivative.

Problem 2

Considering Ω_T as a cylinder, the parabolic boundary is the union of the bottom and the sides of this cylinder, or in other words, the boundary except the top. In symbols: $(\{0\}\times\Omega)\cup([0,T]\times\partial\Omega)$.

At a maximum point in the given set, we must have $u_t \ge 0$, $\nabla u = 0$, $\Delta u \le 0$ (in most cases, we will have $u_t = 0$, but at the top end $t = T$, we can only guarantee the stated inequality). Together with $u^2 \ge 0$ these imply $u_t - \Delta u - |\nabla u|^2 + 1 + u^2 \ge 1$, which contradicts the given equation.

Problem 3

The function on the right hand side (let us call it $u(\boldsymbol{x})$) is harmonic for $\boldsymbol{x} \ne \boldsymbol{0}$. It has the value A when $|\boldsymbol{x}| = a$, and B when $|\boldsymbol{x}| = b$. Thus by assumption $v \le u$ on $\partial\Omega$, and so by the maximum principle $v \le u$ in Ω as well.

Problem 4

Consider any function $v \in C_c^\infty(\Omega)$. If u is a minimiser for $\mathcal{L}$, then the t derivative of $\mathcal{L}[u+tv]$ should vanish at $t = 0$:

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}[u+tv]\Big|_{t=0} = \int_\Omega\bigg(\frac{\nabla u\cdot\nabla v}{\sqrt{1+|\nabla u(\boldsymbol{x})|^2}} + \partial_u V(\boldsymbol{x},u)v\bigg)\,\mathrm{d}^n\boldsymbol{x}$$
$$= \int_\Omega\bigg(-\nabla\cdot\frac{\nabla u}{\sqrt{1+|\nabla u(\boldsymbol{x})|^2}} + \partial_u V(\boldsymbol{x},u)\bigg)v\,\mathrm{d}^n\boldsymbol{x},$$

and so u must satisfy the equation

$$-\nabla\cdot\frac{\nabla u}{\sqrt{1+|\nabla u(\boldsymbol{x})|^2}} + \partial_u V(\boldsymbol{x},u) = 0.$$

Problem 5

a. The shock speed s is given by the Rankine–Hugoniot condition: $q_r - q_\ell = s(u_r - u_\ell)$ where $q = u - u^2$ is the flux function of the problem. This gives the shock speed

$$s = \frac{(u_r - u_r^2) - (u_\ell - u_\ell^2)}{u_r - u_\ell} = 1 - (u_r + u_\ell).$$

The characteristic speed of the problem is $1 - 2u$. The assumption $u_\ell < u_r$ yields to $1 - 2u_\ell > s > 1 - 2u_r$, i.e., the characteristic speed on the left is greater than on the right, and the shock speed lies in between. The is known as the (Lax) *entropy condition*. It (or something like it) is required for the problem to have unique solutions.

b. The characteristic starting at $x = \xi$ for $t = 0$ has equation

$$x = \xi + c(\xi)t, \qquad c(\xi) = 1 - \frac{2}{\xi^2 + 1} = \frac{\xi^2 - 1}{\xi^2 + 1}.$$

Since $-1 \le c(\xi) < 1$ for all ξ, we find $\xi - t \le x < \xi + t$. In particular $\lim_{\xi \to \pm\infty} x = \pm\infty$ for any fixed t, and it follows by continuity that there exists a solution ξ given t and x. The solution is unique so long as $\partial x / \partial \xi > 0$ for all ξ. This means $1 + c'(\xi)t > 0$. Since

$$c'(\xi) = \frac{4\xi}{(\xi^2 + 1)^2}$$

is bounded, the required inequality $1 + c'(\xi)t > 0$ holds for sufficiently small t.

Summarising, given t and x we need to solve the equation $\xi + c(\xi)t = x$ for ξ. Substituting the formula for $c(\xi)$ and simplifying, we end up with the cubic equation

$$\xi^3 + (t - x)\xi^2 + \xi - x - t = 0.$$

Once we have solved that, we put

$$u(t, x) = \frac{1}{\xi^2 + 1}.$$

c. Look for the minimum value of c'. To this end, differentiate, and get

$$c''(\xi) = \frac{4 - 12\xi^2}{(\xi^2 + 1)^3},$$

from which we find that c' has its minimum value at $\xi_* = -\frac{1}{3}\sqrt{3}$. At that point, $c'(\xi_*) = -\frac{3}{4}\sqrt{3}$, and so the shock arises at time $t_* = -1/c'(\xi_*) = \frac{4}{9}\sqrt{3}$. The shock position at time t^* is $x_* = \xi_* + c(\xi_*)t_* = -\frac{1}{3}\sqrt{3} - \frac{1}{2} \cdot \frac{4}{9}\sqrt{3} = -\frac{5}{9}\sqrt{3}$, and the shock starts out with the characteristic speed $c(\xi_*) = -\frac{1}{2}$.

Problem 6

a. We have $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_k \to \infty$, and the eigenfunctions φ_k form an orthonormal basis for $L^2(\Omega)$ and an orthogonal basis for $H_0^1(\Omega)$. Also, each eigenfunction φ_k is real-valued and infinitely differentiable. The eigenvalue relation interpreted in the weak sense becomes

$$\int_\Omega \nabla\varphi_k \cdot \nabla v \,\mathrm{d}^n\boldsymbol{x} = \lambda_k \int_\Omega \varphi_k v \,\mathrm{d}^n\boldsymbol{x}$$

for all $v \in H_0^1(\Omega)$. With $v = \varphi_k$, this becomes

$$\int_\Omega |\nabla\varphi_k|^2 \,\mathrm{d}^n\boldsymbol{x} = \lambda_k \int_\Omega \varphi_k^2 \,\mathrm{d}^n\boldsymbol{x} = \lambda_k,$$

so that

$$\|\varphi_k\|_{H^1}^2 = \int_\Omega \big(|\varphi_m|^2 + |\nabla\varphi_k|^2\big) \,\mathrm{d}^n\boldsymbol{x} = 1 + \lambda_k.$$

b. Since $g \in H_0^1$, we have

$$\sum_{k=1}^\infty |c_k|^2(1+\lambda_k) = \|g\|_{H^1}^2 < \infty.$$

But then also

$$\sum_{k=1}^\infty |c_k e^{-\lambda_k t}|^2(1+\lambda_k) \le \sum_{k=1}^\infty |c_k|^2(1+\lambda_k) < \infty.$$

(because $\lambda_k > 0$), so that the sum converges in $H_0^1(\Omega)$. Furthermore we find

$$\|u_t - g\|_{H^1}^2 = \sum_{k=1}^\infty |c_k(1-e^{-\lambda_k t})|^2(1+\lambda_k) \le \sum_{k=1}^N |c_k(1-e^{-\lambda_k t})|^2(1+\lambda_k) + \sum_{k=N+1}^\infty |c_k|^2(1+\lambda_k)$$

for any N, since $\lambda_k > 0$. Given $\epsilon > 0$, pick N so the latter sum is less than ϵ. Now the first sum has only a finite number of terms, each of which goes to zero as $t \to 0$, so for sufficiently small t, the first sum is also less than ϵ. Thus the entire sum becomes less than 2ϵ, and the stated convergence is proved.

c. We find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\int_\Omega u_t v \,\mathrm{d}^n\boldsymbol{x} + \int_\Omega \nabla u_t \cdot \nabla v \,\mathrm{d}^n\boldsymbol{x} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=1}^\infty c_k e^{-\lambda_k t}\int_\Omega \varphi_k v \,\mathrm{d}^n\boldsymbol{x} + \sum_{k=1}^\infty c_k e^{-\lambda_k t}\int_\Omega \nabla\varphi_k \cdot \nabla v \,\mathrm{d}^n\boldsymbol{x} \\ &= -\sum_{k=1}^\infty c_k\lambda_k e^{-\lambda_k t}\int_\Omega \varphi_k v \,\mathrm{d}^n\boldsymbol{x} + \sum_{k=1}^\infty c_k e^{-\lambda_k t}\lambda_k\int_\Omega \varphi_k v \,\mathrm{d}^n\boldsymbol{x} = 0.\end{aligned}$$

where we took the differentiation into the first sum (justifiable by the fact that the resulting sum is uniformly convergent) and used the weak formulation of $-\Delta\varphi_k = \lambda_k\varphi_k$ in the second sum.