Problem 1 If the non-constant function u satisfies the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = 0$$

in a bounded region Ω, show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2 \,\mathrm{d}^n\boldsymbol{x} < 2\int_{\partial\Omega} u\frac{\partial u}{\partial n}\,\mathrm{d}S.$$

State the regularity conditions on u and Ω needed for your proof.

(The derivative in the integral on the right is the normal derivative.)

Problem 2 Assume Ω to be a bounded domain in $\mathbb{R}^n$, and that $T > 0$. Write $\Omega_T = (0,T)\times\Omega$. What is the parabolic boundary of Ω_T?

Assume that $u \in C(\overline{\Omega_T}) \cap C^2(\Omega_T)$ satisfies the equation

$$\frac{\partial u}{\partial t} - \Delta u - |\nabla u|^2 + 1 + u^2 = 0.$$

Show that u does not have a maximum in $(0,T]\times\Omega$.

Problem 3 Assume that the function $v \in C(\overline{\Omega}) \cap C^2(\Omega)$ is subharmonic in the region $\Omega = \{\boldsymbol{x} \in \mathbb{R}^n \mid a < |\boldsymbol{x}| < b\}$ with $n > 2$ and $0 < a < b$. Assume further that $v(\boldsymbol{x}) \le A$ when $|\boldsymbol{x}| = a$ and $v(\boldsymbol{x}) \le B$ when $|\boldsymbol{x}| = b$. Show that then

$$v(\boldsymbol{x}) \le \frac{\left(\dfrac{1}{a^{n-2}} - \dfrac{1}{|\boldsymbol{x}|^{n-2}}\right)B + \left(\dfrac{1}{|\boldsymbol{x}|^{n-2}} - \dfrac{1}{b^{n-2}}\right)A}{\dfrac{1}{a^{n-2}} - \dfrac{1}{b^{n-2}}} \qquad \text{for all } \boldsymbol{x} \in \Omega.$$

Problem 4 Derive a partial differential equation (the Euler–Lagrange equation) for a function u minimising the functional

$$\mathcal{L}[u] = \int_\Omega \Big(\sqrt{1 + |\nabla u(\boldsymbol{x})|^2} + V\big(\boldsymbol{x}, u(\boldsymbol{x})\big)\Big)\,\mathrm{d}^n\boldsymbol{x}$$

with $u = 0$ on $\partial\Omega$, where Ω is a bounded region in $\mathbb{R}^n$ and V is a given, suitably differentiable function. (For $n = 2$, this is a simple model of a suspended soap film in a force field.)

Problem 5 Consider the traffic equation with initial data:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(u - u^2) = 0, \quad u(0, x) = \frac{1}{x^2 + 1}.$$

a. What is the speed of a shock for the given equation, where the solution jumps from u_ℓ on the left to u_r on the right, and $u_\ell < u_r$? What is the significance of the assumption $u_\ell < u_r$?

b. Show that the problem has a unique classical solution defined for $x \in \mathbb{R}$ and $0 \le t < t_*$, for some $t_* > 0$. Write up a set of algebraic equations whose solution solves the given problem. (Don't try to solve them; they involve a cubic equation.)

c. What is the largest value of t_* so that the classical solution in question **b** exists? A shock will form at time t_*. What is the position x_* and the speed of the shock at the moment of creation?

Problem 6 Assume that $\Omega \subset \mathbb{R}^n$ is a bounded region. Let $\lambda_1, \lambda_2, \ldots$, be the usual eigenvalues of the Laplacian given Dirichlet boundary conditions, with real-valued eigenfunctions $\varphi_1, \varphi_2, \ldots \in H_0^1(\Omega)$ satisfying $-\Delta\varphi_k = \lambda_k\varphi_k$ and $\|\varphi_k\|_2 = 1$.

a. The above description of the eigenvalues and eigenfunctions is incomplete. Briefly fill in the missing details, and show that $\|\varphi_k\|_{H^1}^2 = 1 + \lambda_k$.

Now assume that $g \in H_0^1(\Omega)$, and put

$$u_t = \sum_{k=1}^{\infty} c_k e^{-\lambda_k t}\varphi_k \qquad \text{where } t > 0,\ c_k = \int_\Omega g\varphi_k \, \mathrm{d}^n\boldsymbol{x}.$$

(*Note* that the index t here does not signify a partial derivative.)

b. Show that the sum above converges in $H_0^1(\Omega)$, and that $\lim_{t\to 0} \|u_t - g\|_{H^1} = 0$.

c. Show that u_t is a weak solution to the heat equation in the sense that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_t v \, \mathrm{d}^n\boldsymbol{x} + \int_\Omega \nabla u_t \cdot \nabla v \, \mathrm{d}^n\boldsymbol{x} = 0$$

for all $v \in H_0^1(\Omega)$.