TMA4305 Partial differential equations 2018-12-11

Solution

Problem 1

The characteristic equations will be $x' = 1$, $y' = 2ux$, $u' = 0$. Thanks to the first equation, we may simply use x as the parameter on each characteristic, so the derivatives become d/dx. The final equation says u is constant on each characteristic, so the y equation integrates into $y = ux^2 + C$ for some constant C.

Putting $x = \xi$, $y = 0$, and $u = \xi^{-1}$ (from the given initial data) gives $0 = \xi^{-1}\xi^2 + C$, so $C = -\xi$ for the characteristic passing through $(\xi, 0)$. Also, $u = \xi^{-1}$ on this characteristic, so we must have $y = ux^2 - u^{-1}$.

Rewrite this as $u^2x^2 - uy - 1 = 0$, with the solutions

$$u = \frac{y \pm \sqrt{y^2 + 4x^2}}{2x^2}.$$

Setting $y = 0$ and $x > 0$ in this expression and comparing with the initial data reveals that we must choose the positive sign, so

$$u = \frac{y + \sqrt{y^2 + 4x^2}}{2x^2}.$$

At the outset, this appears to be good only for $x > 0$, as there seems to be a singularity at $x = 0$, and initial data is given only for $x > 0$. The apparent singularity is real enough for $y > 0$; but for $y < 0$, we can get rid of it, by multiplying and dividing by $\sqrt{y^2 + 4x^2} - y$, resulting in the better expression

$$u = \frac{4x^2}{2x^2\bigl(\sqrt{y^2 + 4x^2} - y\bigr)} = \frac{2}{\sqrt{y^2 + 4x^2} - y},$$

valid for all $(x, y) \in \mathbb{R}^2$ except the positive y axis.

Problem 2

a. We find

$$u(t, x) = \frac{1}{2}\int_{x-t}^{x+t} \frac{a}{\pi(1 + a^2x^2)} = \frac{1}{2\pi}\Bigl(\operatorname{atan}\bigl(a(x + t)\bigr) - \operatorname{atan}\bigl(a(x - t)\bigr)\Bigr).$$

b. The two atan expressions in the solution converge to the same limit $\pm\pi/2$ if the two values $x \pm t$ have the same sign. Thus the solution goes to zero. The signs are different if $-t < x < t$; the result is

$$W(t, x) = \begin{cases} 1/2 & \text{if } -t < x < t, \\ 0 & \text{if } x < -t \text{ or } x > t \end{cases}$$

(and $1/4$ if $x = \pm t$, but that is less important). We recognize this as the *wave kernel*, and named it accordingly.

The function g_a is an approximate delta in the limit $a \to \infty$; thus we would like to say that the wave kernel solves the initial value problem for the wave equation with $u|_{t=0} = 0$ and $u_t|_{t=0} = \delta$.

Problem 3

u is a distribution because the function

$$x \mapsto \begin{cases} 2x^{-1/2} & x > 0 \\ 0 & x \le 0 \end{cases}$$

is locally integrable.

The classical derivative of the function is $-x^{-3/2}$ for $x > 0$. The distributional derivative is given by ($\delta \searrow 0$ means $\delta \to 0$ from above)

$$\begin{aligned}(u', \psi) = -(u, \psi') &= -\int_0^\infty 2x^{-1/2}\psi'(x)\,\mathrm{d}x = -\lim_{\delta\searrow 0}\int_\delta^\infty 2x^{-1/2}\psi'(x)\,\mathrm{d}x \\ &= -\lim_{\delta\searrow 0}\Bigl(-2\delta^{-1/2}\bigl(\psi(\delta)-\psi(0)\bigr) + \int_\delta^\infty x^{-3/2}\bigl(\psi(x)-\psi(0)\bigr)\,\mathrm{d}x\Bigr) \\ &= \int_0^\infty x^{-3/2}\bigl(\psi(0)-\psi(x)\bigr)\,\mathrm{d}x.\end{aligned}$$

In the second line, we performed a partial integration, picking $\psi(x)-\psi(0)$ as an antiderivative of $\psi'(x)$, in order to better deal with the singularity at $x = 0$. We also used $\delta^{-1/2}\bigl(\psi(\delta)-\psi(0)\bigr) = \delta^{-1/2}O(\delta) = O(\delta^{1/2}) \to 0$ when $\delta \searrow 0$, and similarly $x^{-3/2}\bigl(\psi(x)-\psi(0)\bigr) = O(x^{-1/2})$ is integrable near $x = 0$. Last, but not least, we rely the integrability $x^{-3/2}$ “at infinity”, i.e., that $\int_\delta^\infty x^{-3/2}\,\mathrm{d}x$ converges.

Since the answer was given, we could also have started with the given solution and done the above calculation in reverse; but the direct approach is more satisfying.

Problem 4

The Cauchy–Schwartz inequality implies $|\gamma(v)| \le \|g\|_2\|v\|_2 \le \|g\|_2\|v\|_{H^1}$; thus γ is bounded.

Taking complex conjugates, we have $\langle u, v\rangle_{H^1} = \langle g, v\rangle$ for all $v \in H^1_0(\Omega)$. Written out in full, that is

$$\int_\Omega \bigl(u\bar{v} + \nabla u \cdot \nabla \bar{v}\bigr)\,\mathrm{d}^n\boldsymbol{x} = \int_\Omega g\bar{v}\,\mathrm{d}^n\boldsymbol{x}.$$

Replacing v by $\bar{v}$, and hence $\bar{v}$ by v, we see that u is a weak solution to the equation

$$u - \Delta u = g.$$

Rewriting this as $\Delta u = u - g$, it follows from a known result that if $u - g \in H^m$, then $u \in H^{m+2}$. But we know *a priori* that $u \in H^1$, so repeated application of this result shows that if $g \in H^m$, then $u \in H^{m+2}$. The Sobolev embedding theorem states that $H^m(\Omega) \subset C^k(\Omega)$ if $m > k + n/2$. Thus we require $m + 2 > 2 + n/2$, that is, $m > n/2$, in order to guarantee that $u \in C^2$.

The solutions continue on the next page ...

Problem 5

Multiplying by u_t puts the equation on the form

$$\left(\tfrac{1}{2}u_t^2\right)_t + au_t^2 + \left(\tfrac{1}{2}bu^2\right)_t - c^2u_tu_{xx} = 0.$$

Since $a > 0$, we immediately get

$$\left(\tfrac{1}{2}u_t^2 + \tfrac{1}{2}bu^2\right)_t - c^2u_tu_{xx} \le 0.$$

To deal with the final term, note that $(u_tu_x)_x = u_tu_{xx} + u_{xt}u_x = u_tu_{xx} + (\frac{1}{2}u_x^2)_t$, so the above inequality becomes

$$\left(\tfrac{1}{2}u_t^2 + \tfrac{1}{2}bu^2 + \tfrac{1}{2}c^2u_x^2\right)_t - \left(u_tu_x\right)_x \le 0.$$

Integrating this over the interval $[0, L]$, and taking the time derivative outside, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \left(\tfrac{1}{2}u_t^2 + \tfrac{1}{2}bu^2 + \tfrac{1}{2}c^2u_x^2\right)\mathrm{d}x \le u_t(0,L)u_x(0,L) - u_t(0,0)u_x(0,0).$$

If u solves the problem with homogeneous data, i.e., initial data and boundary data are all zero, then the integral on the left is zero initially, and the right hand side is zero because $u = 0$ on the boundary implies $u_t = 0$ on the boundary. Thus the energy integral is non-increasing. Since each term of the integrand is nonnegative, the energy integral is zero for all $t > 0$. This immediately implies $u = 0$.

Since the equation and boundary conditions are linear, this implies that the general problem has a unique solution (if it has one at all).

Problem 6

a. Use the divergence theorem:

$$\int_{\mathbb{S}} g\,\mathrm{d}S = \int_{\mathbb{S}} \frac{\partial u}{\partial n}\,\mathrm{d}S = \int_{\mathbb{S}} \boldsymbol{n}\cdot\nabla u\,\mathrm{d}S = \int_{\mathbb{B}} \nabla\cdot\nabla u\,\mathrm{d}^n\boldsymbol{x} = \int_{\mathbb{B}} \Delta u\,\mathrm{d}^n\boldsymbol{x} = 0.$$

b. This is an immediate consequence of the mean value property for harmonic functions.

c. There is an apparent singularity in the integral at $t = 0$, but it is not really singular, since $v(\mathbf{0}) = 0$ and v is differentiable (in fact, it is C^∞). So we can differentiate under the integral sign, and get

$$\Delta u(\boldsymbol{x}) = \int_0^1 t\,\Delta v(t\boldsymbol{x})\,\mathrm{d}t = 0$$

(the chain rule produces a factor t^2 when taking the Laplacian of the integrand with respect to $\boldsymbol{x}$), since v is harmonic.

The normal derivative at $\boldsymbol{x} \in \mathbb{S}$ is

$$\frac{\partial u}{\partial n} = \boldsymbol{x}\cdot\nabla u = \int_0^1 \boldsymbol{x}\cdot\nabla v(t\boldsymbol{x})\,\mathrm{d}t = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}v(t\boldsymbol{x})\,\mathrm{d}t = v(\boldsymbol{x}) - v(\mathbf{0}) = g(\boldsymbol{x})$$

(again, the chain rule produced a factor t when taking the gradient of $u(t\boldsymbol{x})$ with respect to $\boldsymbol{x}$).