NTNU
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Department of Mathematical Sciences

Examination paper for **TMA4305 Partial differential equations**

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Examination time (from–to): 09:00–13:00

Permitted examination support material: Code C: One yellow A4-sized sheet of paper stamped by the Department of Mathematical Sciences. On this sheet the student may write whatever is desired. Specified, simple calculator allowed.

Language: English

Number of pages: 3

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave
Originalen er:
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Date Signature

Problem 1 Solve the problem

$$\frac{\partial u}{\partial x} + 2ux\frac{\partial u}{\partial y} = 0, \qquad u(x,0) = \frac{1}{x} \quad \text{for } x > 0,$$

indicating in which region the solution is valid.

Problem 2

a. Compute d'Alembert's solution of the initial value problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= 0 && \text{for } x \in \mathbb{R} \text{ and } t > 0, \\ \left.\begin{aligned} u(0,x) &= 0 \\ \frac{\partial u}{\partial t}(0,x) &= g_a(x) := \frac{a}{\pi(1+a^2x^2)} \end{aligned}\right\} && \text{for } x \in \mathbb{R}, \end{aligned}$$

where $a > 0$.

b. What is the limit as $a \to \infty$ of the solution found above? Does the answer look familiar? Explain briefly. *Hint*: What can you say about the functions g_a in the limit $a \to \infty$?

Problem 3 The distribution u is defined by

$$(u, \psi) = \int_0^\infty \frac{2\psi(x)}{\sqrt{x}}\,\mathrm{d}x$$

for any test function $\psi \in C_c^\infty(\mathbb{R})$. Explain briefly why this does in fact define a distribution, and show that the distributional derivative of u satisfies

$$(u', \psi) = \int_0^\infty \frac{\psi(0) - \psi(x)}{x^{3/2}}\,\mathrm{d}x.$$

The problem set continues on the next page ...

Problem 4 Let $\Omega \subset \mathbb{R}^n$ be a bounded region. You are reminded that the Sobolev space $H^1_0(\Omega)$ is a Hilbert space whose norm is defined by

$$\|u\|_{H^1} = \sqrt{\int_\Omega \Big(|u|^2 + |\nabla u|^2\Big)\,\mathrm{d}^n\boldsymbol{x}}.$$

Let $g \in L^2(\Omega)$. Show that $\gamma(v) = \langle v, g\rangle$ (where the angle brackets denote the ordinary L^2 inner product) is a bounded linear functional on $H^1_0(\Omega)$. It follows that there exists some $u \in H^1_0(\Omega)$ so that

$$\langle v, u\rangle_{H^1} = \langle v, g\rangle$$

for all $v \in H^1_0(\Omega)$. Here the expression on the left is the inner product in $H^1(\Omega)$. (This is the Riesz representation theorem; you do not need to justify its use beyond the boundedness of γ.)

The function u solves a partial differential equation in the weak sense. Identify this equation. Assuming $g \in H^m(\Omega)$, what can you say about the regularity of u? In particular, how large must m be before you can be certain that $u \in C^2(\Omega)$?

Problem 5 The *telegraph equation* is

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + bu - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

where a, b and c are positive constants. Use an energy method to show that any classical solution of the equation in the domain $t > 0$, $x \in (0, L)$ is unique, given initial and boundary data

$$\left.\begin{aligned} u(0,x) &= g(x) \\ \frac{\partial u}{\partial t}(0,x) &= h(x) \end{aligned}\right\}\text{for } x \in (0,L),$$
$$\left.\begin{aligned} u(t,0) &= g_l(t) \\ u(t,L) &= g_r(t) \end{aligned}\right\}\text{for } t > 0.$$

Hint: Begin by multiplying the equation by $\partial u/\partial t$.

The problem set continues on the next page ...

Problem 6 In this problem, we write $\mathbb{B} = \{\,\boldsymbol{x} \in \mathbb{R}^n \mid |\boldsymbol{x}| < 1\,\}$ for the unit ball in $\mathbb{R}^n$, and $\mathbb{S} = \partial\mathbb{B} = \{\,\boldsymbol{x} \in \mathbb{R}^n \mid |\boldsymbol{x}| = 1\,\}$ for its boundary, the unit sphere. (Usually, one writes $\mathbb{B}^n$ and $\mathbb{S}^{n-1}$.) The *Neumann problem* on the unit ball is the problem

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \mathbb{B}, \\ \frac{\partial u}{\partial n} &= g \quad \text{on } \mathbb{S}, \end{aligned} \tag{N}$$

where $g \in C(\mathbb{S})$ is a given function.

a. Show that if (N) has a solution $u \in C^2(\mathbb{B}) \cap C(\overline{\mathbb{B}})$, then

$$\int_{\mathbb{S}} g \, \mathrm{d}S = 0. \tag{$*$}$$

b. Assume that g satisfies ($*$). Let v be the solution of the *Dirichlet problem*

$$\begin{aligned} \Delta v &= 0 \quad \text{in } \mathbb{B}, \\ v &= g \quad \text{on } \mathbb{S}. \end{aligned} \tag{D}$$

Show that $v(\mathbf{0}) = 0$.

c. With v as above, put

$$u(\boldsymbol{x}) = \int_0^1 \frac{v(t\boldsymbol{x})}{t} \, \mathrm{d}t \qquad \text{for } \boldsymbol{x} \in \overline{\mathbb{B}}.$$

Show that u solves (N).