TMA4305 Partial differential equations 2017-11-29

Solution

Problem 1

a. We do both cases at once, always with $\mp$ being the sign opposite $\pm$:

$$\frac{\mathrm{d}E_\pm}{\mathrm{d}t} = \frac{1}{2}\int_0^t (u_t \mp u_x)(u_{tt} \mp u_{xt})\,\mathrm{d}x = \frac{1}{2}\int_0^t (u_t \mp u_x)(u_{xx} \mp u_{xt})\,\mathrm{d}x = \mp\frac{1}{4}\int_0^t \frac{\mathrm{d}}{\mathrm{d}x}(u_t \mp u_x)^2\,\mathrm{d}x$$
$$= \mp\Big[(u_t \mp u_x)^2\Big]_{x=0}^{x=1} = \mp\big(e_\pm(t,1) - e_\pm(t,0)\big).$$

ALTERNATIVE SOLUTION: (Here using ∂_t as a shorthand for $\partial/\partial t$, etc.) Note that $(\partial_t \pm \partial_x)(u_t \mp u_x) = u_{tt} - u_{xx} = 0$, so that $u_t \mp u_x$, and therefore $e_\pm$, is a function of $x \mp t$. Say, $e_\pm(t,x) = \tilde{e}_\pm(x \mp t)$: Then

$$E_\pm(t) = \int_0^1 \tilde{e}_\pm(x \mp t)\,\mathrm{d}x = \int_{\mp t}^{1\mp t} \tilde{e}_\pm(s)\,\mathrm{d}s,$$

and differentiation with respect to t yields the desired result by the fundamental theorem of calculus.

b. Start with the integral:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t e(t,x)\,\mathrm{d}x = \frac{\mathrm{d}E_+}{\mathrm{d}t} + \frac{\mathrm{d}E_-}{\mathrm{d}t} = \big(e_+(t,0) - e_-(t,0)\big) - \big(e_+(t,1) - e_-(t,1)\big).$$

Next, get

$$e_+(t,0) - e_-(t,0) + \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{a}{2}u(t,0)^2\Big) = -u_t(t,0)u_x(t,0) + au(t,0)u_t(t,0) = 0$$

by the given boundary condition at $x = 0$. Similarly,

$$-\big(e_+(t,1) - e_-(t,1)\big) + \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{b}{2}u(t,1)^2\Big) = u_t(t,1)u_x(t,1) + bu(t,1)u_t(t,1) = 0.$$

Combining the results of the above three calculations yields the desired result.

The uniqueness result follows because the problem, including the initial and boundary conditions, is linear. The difference of two solutions will then solve the homogeneous problem. The energy of the difference will be zero initially, and hence zero forever, so the difference remains zero – because the t derivative must vanish, and integrating from $t = 0$ completes the argument.

Problem 2

Let λ_1 be the smallest eigenvalue, and u_1 the corresponding (real) eigenfunction. When u is a real eigenfunction with an eigenvalue $\lambda \neq \lambda_1$, we know that $u \perp u_1$:

$$\int_\Omega uu_1\,\mathrm{d}^n\boldsymbol{x} = 0$$

(without any need for a complex conjugate, since u and u_1 are real-valued). The integrand cannot be identically zero, since u is never zero in Ω. Therefore, the integrand must be of both signs, and hence so must u, since u_1 is not.

Problem 3

a. Put $E(t) = \int_\Omega \frac{1}{2}u^2 \, \mathrm{d}^n\boldsymbol{x}$, and compute

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_\Omega uu_t \, \mathrm{d}^n\boldsymbol{x} = \int_\Omega u\nabla\cdot(c\nabla u)\,\mathrm{d}^n\boldsymbol{x} = \int_{\partial\Omega} \boldsymbol{\nu}\cdot(uc\nabla u)\,\mathrm{d}S - \int_\Omega \nabla u\cdot(c\nabla u)\,\mathrm{d}^n\boldsymbol{x}$$

where the first integral vanishes because $c = 0$ on $\partial\Omega$, so we get

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\int_\Omega c\,|\nabla u|^2\,\mathrm{d}^n\boldsymbol{x} \le 0$$

because $c \ge 0$. Thus $E(t)$ in non-increasing. Since $E(t) \ge 0$ always, and the initial condition gives $E(0) = 0$, we must have $E(t) = 0$ for all t, and so $u = 0$.

b. The weak maximum principle: Given a bounded domain Ω and $T > 0$, let $\Omega_T = (0,T)\times\Omega$, and $\Gamma = (\{0\}\times\Omega)\cup([0,T]\times\partial\Omega)$ (the parabolic boundary). Then any solution $u \in C^2(\overline{\Omega_T})$ of the heat equation $u_t - \Delta u = 0$ achieves its maximum on Γ.

The same holds when u solves the equation from part **a**: Let $\epsilon > 0$, and let $v(t,\boldsymbol{x}) = u(t,\boldsymbol{x}) - \epsilon t$. Then $v_t - \nabla\cdot(c\nabla v) = -\epsilon < 0$. If v has a maximum at some point $(t_0,\boldsymbol{x}_0) \in \Omega_T\setminus\Gamma$, then $\nabla v(t_0,\boldsymbol{x}_0) = 0$, and so $\nabla\cdot(c\nabla v) = c\,\Delta v + \nabla c\cdot\nabla v = c\,\Delta v \le 0$ at $(t_0,\boldsymbol{x}_0)$. Also, $v_t \ge 0$ (with equality if $t < T$), so $v_t - \nabla\cdot c\nabla v \ge 0$ at $(t_0,\boldsymbol{x}_0)$, which is a contradiction. But v, being a continuous function, has a maximum somewhere on the compact set $\overline{\Omega_T}$, so this maximum must occur in Γ. We conclude

$$u = v + \epsilon t \le \max_\Gamma v + \epsilon T \le \max_\Gamma u + \epsilon T.$$

Now letting $\epsilon \to 0$, we conclude $u \le \max_\Gamma u$.

Problem 4

a. First, Δu is subharmonic because $\Delta(\Delta u) \ge 0$. Therefore, $\Delta u \le 0$ in Ω because of the subharmonicity and the assumption $\Delta u \le 0$ on $\partial\Omega$ (from the weak maximum principle applied to $v = \Delta u$). Thus u is superharmonic. It follows, this time from the weak minimum principle, that $u \ge 0$ on Ω. Finally, the strong minimum principle asserts that if u has any zero in Ω, then u is constant. Since it is not, then $u > 0$ in Ω.

b. Note that u achieves its maximum and minimum in $\overline{\Omega}$, because of continuity and compactness. If the maximum is greater than 1, then it is achieved in Ω, and $\Delta u = u^3 - u > 0$ at the maximum point. But this cannot happen, since $\Delta u \le 0$ at a maximum. Therefore $u \le 1$ in Ω.

A similar argument shows that $u \ge -1$. A quicker way is to note that $-u$ satisfies the same conditionss as u itself (with different boundary values, but still between -1 and 1), so the first part of the argument shows that $-u \le 1$.

Problem 5

a. The quasilinear form of the equation is

$$h_t + 2h^{1/2}h_x = 0,$$

so the characteristic speed associated with h is $2h^{1/2}$. Therefore, the characteristic starting at $(t,x) = (0,\xi)$ has speed $2(1-\xi)^{1/2}$ if $0 \le \xi \le 1$, 2 of $\xi < 0$, and 0 if $\xi > 1$. The equation of this characteristic, then, is

$$\begin{cases} x = \xi + 2t & \text{if } \xi < 0, \\ x = \xi + 2(1-\xi)^{1/2}t & \text{if } 0 \le \xi \le 1, \\ x = \xi & \text{if } \xi > 1. \end{cases}$$

We can detect the collision of characteristics by taking the derivative $\partial_\xi x$ in the above equations. When this derivative becomes ≤ 0, characteristics collide. For $0 < \xi < 1$, we find $\partial_\xi x = 1 - (1-\xi)^{-1/2}t$, and $\partial_\xi x \le 0$ if and only if $t \ge (1-\xi)^{1/2}$. Since $(1-\xi)^{1/2} \to 0$ when $\xi \to 1$, characteristics start colliding near $\xi = 1$ at time 0.

b. In the specified region, the solution is given by the characteristics emanating from $(t, x) = (0, \xi)$ with $0 < \xi < 1$. On one such characteristic, with the equation $x = \xi + 2(1-\xi)^{1/2}t$, we know that $h = 1 - \xi$, so we may write $x = 1 - h + 2h^{1/2}t$ and solve for h without solving for ξ first. Here, it is best to treat $h^{1/2}$ as the unknown at first. We get, in order:

$$\begin{aligned} h - 2th^{1/2} &= 1 - x \\ \bigl(h^{1/2} - t\bigr)^2 &= 1 - x + t^2 \\ h^{1/2} &= t \pm (1 - x + t^2)^{1/2}. \end{aligned}$$

Looking at the initial condition and setting $t = 0$, it is clear that we must pick the positive sign:

$$h = \bigl(t + (1 - x + t^2)^{1/2}\bigr)^2.$$

c. The flux function for the problem is $f(h) = \frac{4}{3}h^{3/2}$. The solution on the left side of the shock, $h(t, \sigma(t)^-)$, is given by the solution found in **b**, while the solution on the right is zero. The Rankine–Hugoniot condition becomes

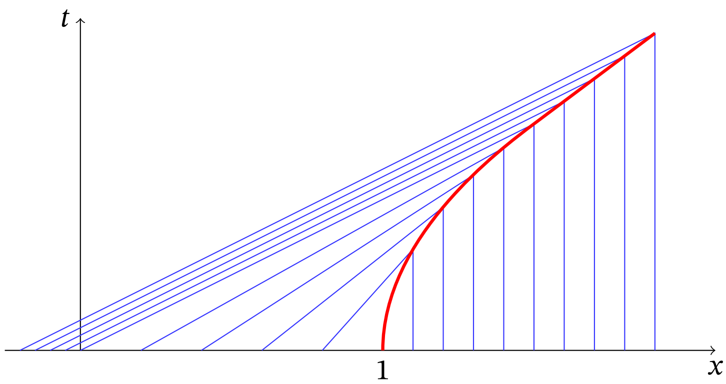
$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{f\bigl(h(t,\sigma(t)^-)\bigr) - f\bigl(h(t,\sigma(t)^+)\bigr)}{h(t,\sigma(t)^-) - h(t,\sigma(t)^+)} = \frac{\frac{4}{3}\bigl(h(t,\sigma(t)^-)\bigr)^{3/2}}{h(t,\sigma(t)^-)} = \tfrac{4}{3}\bigl(h(t,\sigma(t)^-)\bigr)^{1/2},$$

and finally inserting the value of h from **b**, we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \tfrac{4}{3}\bigl(t + (1 - \sigma(t) + t^2)^{1/2}\bigr).$$

This has the solution $\sigma(t) = 1 + \frac{8}{9}t^2$, which also fits the initial condition $\sigma(0) = 1$. This remarkably simple solution is *not easy to discover*,[1] which is why we did not ask for it.

Even without knowing the exact solution for $\sigma(t)$, it should be clear that $\sigma'(0) = 0$, and that $\sigma'(t)$ is an increasing function, since the $h(\sigma(t)^-)$ increases with time. See the picture below, where the curve $x = \sigma(t)$ is drawn in red, and the characteristics in blue. Note that, at some point – to be precise, at $(t, x) = (\frac{3}{4}, \frac{3}{2})$ – the characteristics from the constant initial data at $\xi < 0$ reach the characteristic curve, so its speed becomes constant.



[1]It was found almost by accident, *after* this problem set was created!