# TMA4305 Partial differential equations 2017-11-29

## *Solution*

### Problem 1

**a.** We do both cases at once, always with $\mp$ being the sign opposite $\pm$:

$$\frac{\mathrm{d}E_\pm}{\mathrm{d}t} = \frac{1}{2}\int_0^t (u_t \mp u_x)(u_{tt} \mp u_{xt})\,\mathrm{d}x = \frac{1}{2}\int_0^t (u_t \mp u_x)(u_{xx} \mp u_{xt})\,\mathrm{d}x = \mp\frac{1}{4}\int_0^t \frac{\mathrm{d}}{\mathrm{d}x}(u_t \mp u_x)^2\,\mathrm{d}x$$
$$= \mp\Big[(u_t \mp u_x)^2\Big]_{x=0}^{x=1} = \mp\big(e_\pm(t,1) - e_\pm(t,0)\big).$$

ALTERNATIVE SOLUTION: (Here using $\partial_t$ as a shorthand for $\partial/\partial t$, etc.) Note that $(\partial_t \pm \partial_x)(u_t \mp u_x) = u_{tt} - u_{xx} = 0$, so that $u_t \mp u_x$, and therefore $e_\pm$, is a function of $x \mp t$. Say, $e_\pm(t,x) = \tilde{e}_\pm(x \mp t)$: Then

$$E_\pm(t) = \int_0^1 \tilde{e}_\pm(x \mp t)\,\mathrm{d}x = \int_{\mp t}^{1\mp t} \tilde{e}_\pm(s)\,\mathrm{d}s,$$

and differentiation with respect to $t$ yields the desired result by the fundamental theorem of calculus.

**b.** Start with the integral:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t e(t,x)\,\mathrm{d}x = \frac{\mathrm{d}E_+}{\mathrm{d}t} + \frac{\mathrm{d}E_-}{\mathrm{d}t} = \big(e_+(t,0) - e_-(t,0)\big) - \big(e_+(t,1) - e_-(t,1)\big).$$

Next, get

$$e_+(t,0) - e_-(t,0) + \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{a}{2}u(t,0)^2\Big) = -u_t(t,0)u_x(t,0) + au(t,0)u_t(t,0) = 0$$

by the given boundary condition at $x = 0$. Similarly the boundary condition at $x = 1$ gives

$$-\big(e_+(t,1) - e_-(t,1)\big) + \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{b}{2}u(t,1)^2\Big) = u_t(t,1)u_x(t,1) + bu(t,1)u_t(t,1) = 0.$$

Combining the results of the above three calculations yields the desired result – i.e., That the derivative of the energy

$$\int_0^1 e(t,x)\,\mathrm{d}x + \frac{a}{2}u(t,0)^2 + \frac{b}{2}u(t,1)^2$$

is zero.

In particular, since $a \ge 0$, $b \ge 0$ and $e \ge 0$, it follows that if this energy is zero initially, and hence for all $t$ then $u_t = u_x = 0$ as well, and so $u = 0$ identically. Applying this to the difference of two solutions to the boundary-initial value problem shows uniqueness.

### Problem 2

Let $\lambda_1$ be the smallest eigenvalue, and $u_1$ the corresponding (real) eigenfunction. When $u$ is a real eigenfunction with an eigenvalue $\lambda \ne \lambda_1$, we know that $u \perp u_1$:

$$\int_\Omega uu_1\,\mathrm{d}^n\boldsymbol{x} = 0$$

(without any need for a complex conjugate, since $u$ and $u_1$ are real-valued). The integrand cannot be identically zero, since $u_1$ is never zero in $\Omega$. Therefore, the integrand must be of both signs, and hence so must $u$, since $u_1$ is not.

**Problem 3**

**a.** Put $E(t) = \int_\Omega \frac{1}{2}u^2 \, \mathrm{d}^n\boldsymbol{x}$, and compute

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_\Omega u u_t \, \mathrm{d}^n\boldsymbol{x} = \int_\Omega u\nabla\cdot(c\nabla u)\, \mathrm{d}^n\boldsymbol{x} = \int_{\partial\Omega} \boldsymbol{\nu}\cdot(uc\nabla u)\, \mathrm{d}S - \int_\Omega \nabla u\cdot(c\nabla u)\, \mathrm{d}^n\boldsymbol{x}$$

where the first integral vanishes because $c = 0$ on $\partial\Omega$, so we get

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\int_\Omega c\,|\nabla u|^2 \, \mathrm{d}^n\boldsymbol{x} \le 0$$

because $c \ge 0$. Thus $E(t)$ in non-increasing. Since $E(t) \ge 0$ always, and the initial condition gives $E(0) = 0$, we must have $E(t) = 0$ for all $t$, and so $u = 0$.

**b.** The weak maximum principle: Given a bounded domain $\Omega$ and $T > 0$, let $\Omega_T = (0,T)\times\Omega$, and $\Gamma = \big(\{0\}\times\Omega\big)\cup\big([0,T]\times\partial\Omega\big)$ (the parabolic boundary). Then any solution $u \in C(\overline{\Omega_T})\cap C^2(\overline{\Omega_T})$ of the heat equation $u_t - \Delta u = 0$ achieves its maximum on $\Gamma$.

The same holds when $u$ solves the equation from part **a**: Let $\epsilon > 0$, and let $v(t,\boldsymbol{x}) = u(t,\boldsymbol{x}) - \epsilon t$. Then $v_t - \nabla\cdot(c\nabla v) = -\epsilon < 0$. If $v$ has a maximum at some point $(t_0,\boldsymbol{x}_0) \in \Omega_T\setminus\Gamma$, then $\nabla v(t_0,\boldsymbol{x}_0) = 0$, and so $\nabla\cdot(c\nabla v) = c\,\Delta v + \nabla c\cdot\nabla v = c\,\Delta v \le 0$ at $(t_0,\boldsymbol{x}_0)$. Also, $v_t \ge 0$ (with equality if $t < T$), so $v_t - \nabla\cdot c\nabla v \ge 0$ at $(t_0,\boldsymbol{x}_0)$, which is a contradiction. But $v$, being a continuous function, has a maximum somewhere on the compact set $\overline{\Omega_T}$, so this maximum must occur in $\Gamma$. We conclude

$$u = v + \epsilon t \le \max_\Gamma v + \epsilon T \le \max_\Gamma u + \epsilon T.$$

Now letting $\epsilon \to 0$, we conclude $u \le \max_\Gamma u$.

**Problem 4**

**a.** First, $\Delta u$ is subharmonic because $\Delta(\Delta u) \ge 0$. Therefore, $\Delta u \le 0$ in $\Omega$ because of the subharmonicity and the assumption $\Delta u \le 0$ on $\partial\Omega$ (from the weak maximum principle applied to $v = \Delta u$). Thus $u$ is superharmonic. It follows, this time from the weak minimum principle, that $u \ge 0$ on $\Omega$. Finally, the strong minimum principle asserts that if $u$ has any zero in $\Omega$, then $u$ is constant. Since it is not, then $u > 0$ in $\Omega$.

**b.** Note that $u$ achieves its maximum and minimum in $\overline{\Omega}$, because of continuity and compactness. If the maximum is greater than 1, then it is achieved in $\Omega$, and $\Delta u = u^3 - u > 0$ at the maximum point. But this cannot happen, since $\Delta u \le 0$ at a maximum. Therefore $u \le 1$ in $\Omega$.

A similar argument shows that $u \ge -1$. A quicker way is to note that $-u$ satisfies the same conditions as $u$ itself (with different boundary values, but still between $-1$ and 1), so the first part of the argument shows that $-u \le 1$.

**Problem 5**

**a.** The quasilinear form of the equation is

$$h_t + 2h^{1/2}h_x = 0,$$

so the characteristic speed associated with $h$ is $2h^{1/2}$. Therefore, the characteristic starting at $(t,x) = (0,\xi)$ has speed $2(1-\xi)^{1/2}$ if $0 \le \xi \le 1$, 2 of $\xi < 0$, and 0 if $\xi > 1$. The equation of this characteristic, then, is

$$\begin{cases} x = \xi + 2t & \text{if } \xi < 0,\\ x = \xi + 2(1-\xi)^{1/2}t & \text{if } 0 \le \xi \le 1,\\ x = \xi & \text{if } \xi > 1.\end{cases}$$

We can detect the collision of characteristics by taking the derivative $\partial_\xi x$ in the above equations. When this derivative becomes $\le 0$, characteristics collide. For $0 < \xi < 1$, we find $\partial_\xi x = 1 - (1-\xi)^{-1/2}t$, and $\partial_\xi x \le 0$ if and only if $t \ge (1-\xi)^{1/2}$. Since $(1-\xi)^{1/2} \to 0$ when $\xi \to 1$, characteristics start colliding near $\xi = 1$ at time 0.

**b.** In the specified region, the solution is given by the characteristics emanating from $(t,x)=(0,\xi)$ with $0<\xi<1$. On one such characteristic, with the equation $x=\xi+2(1-\xi)^{1/2}t$, we know that $h=1-\xi$, so we may write $x=1-h+2h^{1/2}t$ and solve for $h$ without solving for $\xi$ first. Here, it is best to treat $h^{1/2}$ as the unknown at first. We get, in order:

$$\begin{aligned} h-2th^{1/2} &= 1-x \\ \big(h^{1/2}-t\big)^2 &= 1-x+t^2 \\ h^{1/2} &= t\pm(1-x+t^2)^{1/2}. \end{aligned}$$

Looking at the initial condition and setting $t=0$, it is clear that we must pick the positive sign:

$$h=\big(t+(1-x+t^2)^{1/2}\big)^2.$$

**c.** The flux function for the problem is $f(h)=\frac{4}{3}h^{3/2}$. The solution on the left side of the shock, $h(t,\sigma(t)^-)$, is given by the solution found in **b**, while the solution on the right is zero. The Rankine–Hugoniot condition becomes

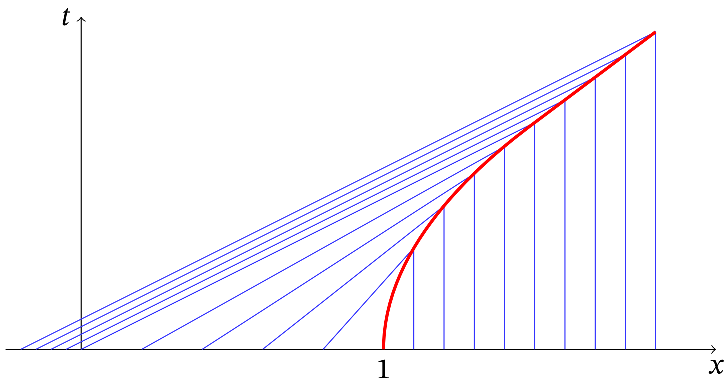
$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}=\frac{f\big(h(t,\sigma(t)^-)\big)-f\big(h(t,\sigma(t)^+)\big)}{h(t,\sigma(t)^-)-h(t,\sigma(t)^+)}=\frac{\frac{4}{3}\big(h(t,\sigma(t)^-)\big)^{3/2}}{h(t,\sigma(t)^-)}=\frac{4}{3}\big(h(t,\sigma(t)^-)\big)^{1/2},$$

and finally inserting the value of $h$ from **b**, we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}=\frac{4}{3}\big(t+(1-\sigma(t)+t^2)^{1/2}\big).$$

This has the solution $\sigma(t)=1+\frac{8}{9}t^2$, which also fits the initial condition $\sigma(0)=1$. This remarkably simple solution is *not easy to discover*,[1] which is why we did not ask for it.

Even without knowing the exact solution for $\sigma(t)$, it should be clear that $\sigma'(0)=0$, and that $\sigma'(t)$ is an increasing function, since the $h(\sigma(t)^-)$ increases with time. See the picture below, where the curve $x=\sigma(t)$ is drawn in red, and the characteristics in blue. Note that, at some point – to be precise, at $(t,x)=(\frac{3}{4},\frac{3}{2})$ – the characteristics from the constant initial data at $\xi<0$ reach the characteristic curve, so its speed becomes constant.




[1]It was found almost by accident, *after* this problem set was created!