NTNU
Norwegian University of
Science and Technology

Department of Mathematical Sciences

# Examination paper for **TMA4305 Partial differential equations**

**Academic contact during examination:** Harald Hanche-Olsen

**Phone:** 7359 3525 / SfB: harald.hanche-olsen@ntnu.no

**Examination date:** Wednesday 29 November 2017

**Examination time (from–to):** 09:00–13:00

**Permitted examination support material:** Code C: One yellow A4-sized sheet of paper stamped by the Department of Mathematical Sciences. On this sheet the student may write whatever is desired. Specified, simple calculator allowed.

**Other information:**
A minor misprint in the original has been corrected (marked in red) in this version.

**Language:** English

**Number of pages:** 3

**Number of pages enclosed:** 0

**Checked by:**

| |
|---|
| **Informasjon om trykking av eksamensoppgave** |
| **Originalen er:** |
| **1-sidig** ☐ **2-sidig** ☒ |
| **sort/hvit** ☒ **farger** ☐ |
| **skal ha flervalgskjema** ☐ |

Date Signature

**Problem 1** This problem uses subscripts for partial derivatives: $u_t = \partial u/\partial t$, etc.

To a $C^2$ function $u$ solving the wave equation $u_{tt} - u_{xx} = 0$, we associate the *energy density* $e = \frac{1}{2}(u_t^2 + u_x^2)$, and also the *right traveling energy density* $e_+ = \frac{1}{4}(u_t - u_x)^2$ and the *left traveling energy density* $e_- = \frac{1}{4}(u_t + u_x)^2$, so that $e = e_+ + e_-$.

**a.** Let $u$ be a solution to the wave equation for $x \in (0,1)$. Put

$$E_\pm(t) = \int_0^1 e_\pm(t,x)\,\mathrm{d}x$$

and show that

$$\frac{\mathrm{d}E_+}{\mathrm{d}t} = e_+(t,0) - e_+(t,1), \qquad \frac{\mathrm{d}E_-}{\mathrm{d}t} = e_-(t,1) - e_-(t,0).$$

**b.** Assume the solution $u$ satisfies boundary conditions

$$u_x(t,0) - au(t,0) = 0, \quad u_x(t,1) + bu(t,1) = 0$$

where $a \geq 0$ and $b \geq 0$ are given constants. Show that

$$\int_0^1 e(t,x)\,\mathrm{d}x + \frac{a}{2}u(t,0)^2 + \frac{b}{2}u(t,1)^2$$

is constant, and that the problem

$$\begin{aligned} u_{tt} - u_{xx} &= f(t,x) && \text{for } x \in (0,1) \text{ and } t > 0 \\ u(0,x) &= g(x) && \text{for } x \in (0,1) \\ u_x - 2au &= h_0(t) && \text{for } x = 0 \text{ and } t > 0 \\ u_x + 2bu &= h_1(t) && \text{for } x = 1 \text{ and } t > 0 \end{aligned}$$

has at most one solution in $C^2\big((0,\infty)\times(0,1)\big) \cap C^1\big([0,\infty)\times[0,1]\big)$ for given functions $f$, $g$, $h_0$, and $h_1$.

**Problem 2** Let $u$ be a real-valued function satisfying $-\Delta u = \lambda u$ in a bounded domain $\Omega$. It is known that if $\lambda$ is the smallest eigenvalue of $-\Delta$, then $u$ is of one sign: Always positive, or always negative. Show that $u$ is not of one sign in $\Omega$ otherwise.

*Hint*: Orthogonality.

**Problem 3** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with piecewise $C^1$ boundary. Assume that a function $c \in C^1([0,T] \times \overline{\Omega})$ is given, with $c > 0$ inside $\Omega$ and $c = 0$ on $\partial\Omega$.

**a.** Show that the initial value problem

$$\frac{\partial u}{\partial t} - \nabla \cdot \big(c(t,\boldsymbol{x})\nabla u\big) = 0 \quad \text{in } (0,T) \times \Omega, \tag{1}$$

$$u = 0 \quad \text{in } \Omega \text{ for } t = 0, \tag{2}$$

has only the trivial solution $u = 0$ in $C^2([0,T] \times \overline{\Omega})$.

*Hint*: Consider $\displaystyle\int_\Omega \tfrac{1}{2}u^2 \, \mathrm{d}^n\boldsymbol{x}$.

**b.** State the weak maximum principle for the ordinary heat equation $u_t - \Delta u = 0$. Show that it holds for equation (1) as well, where we no longer assume (2).

*Hint*: Look at $u(t,\boldsymbol{x}) - \varepsilon t$ where $\varepsilon > 0$, and let $\varepsilon \to 0$.

**Problem 4**

**a.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and assume that $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ is a non-constant function satifying

$$\begin{aligned} \Delta(\Delta u) &\ge 0 \quad \text{in } \Omega, \\ \Delta u &\le 0 \quad \text{on } \partial\Omega, \\ u &\ge 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Show that $u > 0$ in $\Omega$.

*Hint*: Consider the function $v = \Delta u$ first.

**b.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and assume that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies

$$\begin{aligned} -\Delta u &= u - u^3 && \text{in } \Omega, \\ u &= \tfrac{1}{2} && \text{on } \partial\Omega. \end{aligned}$$

Prove that $-1 \le u \le 1$ in $\Omega$.

*Hint*: Assume the opposite, and use a maximum principle.

**Problem 5** Let $h(t,x)$ be the depth of water in a river at time $t$ and position $x$ along the river. After a suitable scaling of the variables, the water depth satisfies the equation

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\tfrac{4}{3}h^{3/2}\right) = 0. \tag{3}$$

**a.** Find the characteristics of equation (3) corresponding to the initial condition

$$h(0,x) = \begin{cases} 1 & \text{if } x < 0, \\ 1 - x & \text{if } 0 \le x \le 1, \\ 0 & \text{if } x > 1, \end{cases}$$

and show that the characteristics start colliding immediately near $x = 1$.

**b.** The weak solution will develop a discontinuity (shock) which can be described as a curve $x = \sigma(t)$ for $t \ge 0$, with $\sigma(0) = 1$. What is the value of $h(t,x)$ in the region $2t < x < \sigma(t)$?

**c.** Write up a differential equation satisfied by $\sigma(t)$. *You do not need to solve it*! Draw a rough sketch showing the characteristics and the shock curve $x = \sigma(t)$.